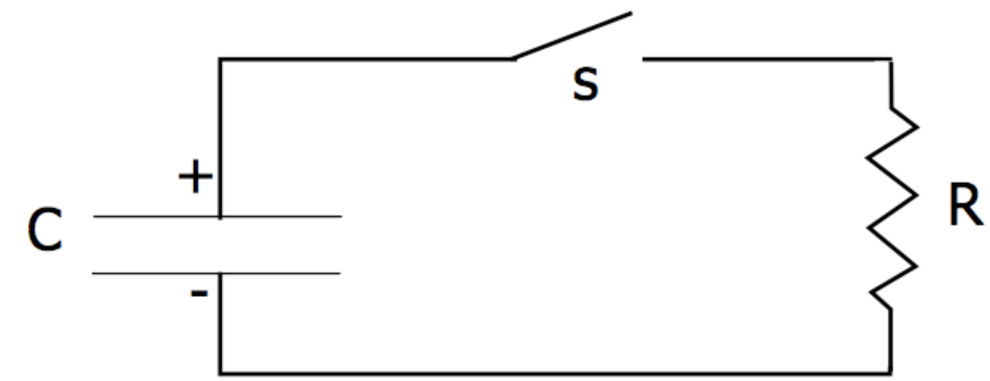
So far we considered only emf and R in the circuits. What if we use capacitors?

Capacitors in circuits

A new way of looking at problems:

- Until now: charges at rest or constant currents
- When capacitors present: currents vary over time



Consider the following situation:

- A capacitor C with charge $Q_0 \rightarrow V_0 = Q_0/C$
- A resistor R in series connected by switch s

What happens when switch s is closed?

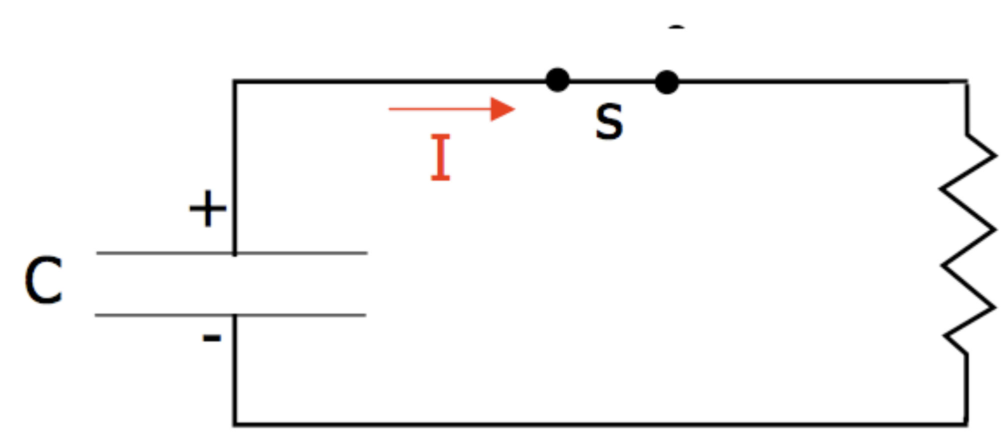
Discharging capacitors: qualitative

Before switch s is closed:

- Difference in potential between C plates: V_0
- No current circulating in the circuit (open)

After switch s is closed:

- Difference in potential between capacitor plates will induce current I
- As I flows, charge difference on capacitor decreases $\rightarrow V_C$ decreases $\rightarrow$ I decreases over time



Discharging capacitors: quantitative

- Apply second Kirchhoff's law:
  - EMF supplied by capacitor C: V=Q/C
    - Note: this is true at any moment in time →Q(t) →V(t)
  - Voltage drop across the resistor: -IR

$$\frac{Q}{C} - IR = 0$$

- Not useful in this form since I=I(Q)
  - I=-dQ/dt (- sign because C is losing charge)

$$\frac{Q}{C} + R\frac{dQ}{dt} = 0$$

- Easy integral yields to exponential decay of the charge:

$$\frac{Q}{C} + R\frac{dQ}{dt} = 0 \Rightarrow \frac{dQ}{Q} = -\frac{1}{RC}dt \Rightarrow \int_0^t \frac{dQ}{Q} = -\frac{1}{RC}\int_0^t dt$$

$$\Rightarrow \ln Q(t) - \ln Q(0) = -\frac{t}{RC} \Rightarrow \ln\frac{Q(t)}{Q_0} = -\frac{t}{RC}$$

$$\Rightarrow \frac{Q(t)}{Q_0} = e^{-\frac{t}{RC}} \Rightarrow Q(t) = Q_0 e^{-\frac{t}{RC}} = Q_0 e^{-\frac{t}{\tau}}$$

τ = RC is called "decay constant" of the circuit

Solution of RC circuit

Solution:

$$Q(t) = Q_0 e^{-\frac{t}{RC}}$$

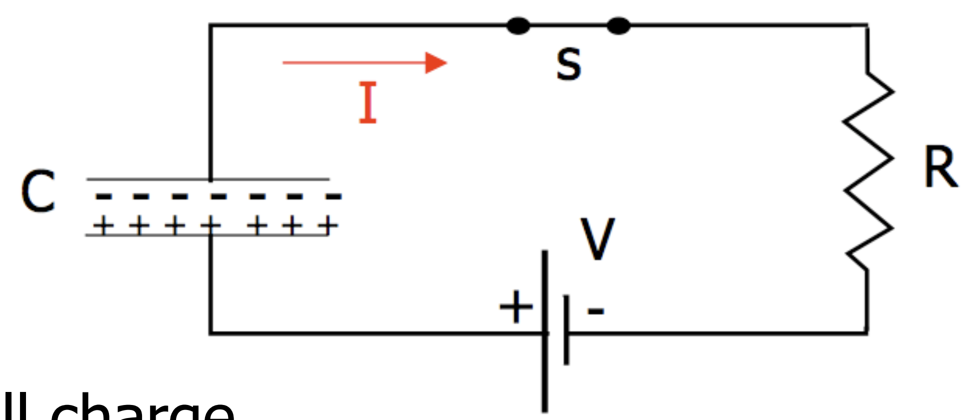
- Exponential decay of charge stored in capacitor
- What are the units of RC?
  - cgs: [R]=statvolts /esu; [C]=esu/statvolt →[RC]=s
  - SI: [R]=V/A; [C]=C/V; A=C/s →[RC]=s
- τ=RC is called "decay constant" of the circuit
  - After a time RC, the charge decreased by 1/e wrt original value

Derive the current:

$$I(t) = -\frac{dQ}{dt} = Q_0 \frac{d}{dt}\left(e^{-\frac{t}{RC}}\right) = \frac{Q_0}{RC} e^{-\frac{t}{RC}}$$

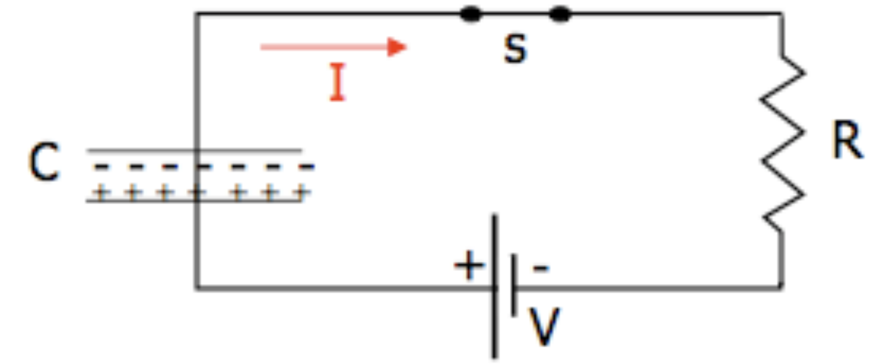
Same exponential decay as for Q(t)

Charging capacitors



- Now 3 elements in circuit: EMF, capacitor and resistor
  - Capacitor starts uncharged
- What happens when switch s is closed?
  - When s is closed, current will suddenly flow and C will charge
  - As C charges, E opposite to EMF builds up and slows down current
  - I(t) stops when V_C reaches V

Charging capacitor: solve the circuit $V - \frac{Q}{C} - IR = 0$



Solve using Kirchhoff's second law:

I(t) = +dQ/dt

+ because the capacitor is now charging!

First order differential equation

$$R\frac{dQ}{dt} + \frac{Q}{C} - V = 0$$

Details of integration

$$R\frac{dQ}{dt} + \frac{Q}{C} - V = 0 \Rightarrow \frac{dQ}{dt} = -\frac{Q - VC}{RC}$$

$$Q' = Q - CV \Rightarrow \frac{dQ'}{dt} = -\frac{Q'}{RC} \Rightarrow \frac{dQ'}{Q'} = -\frac{dt}{RC}$$

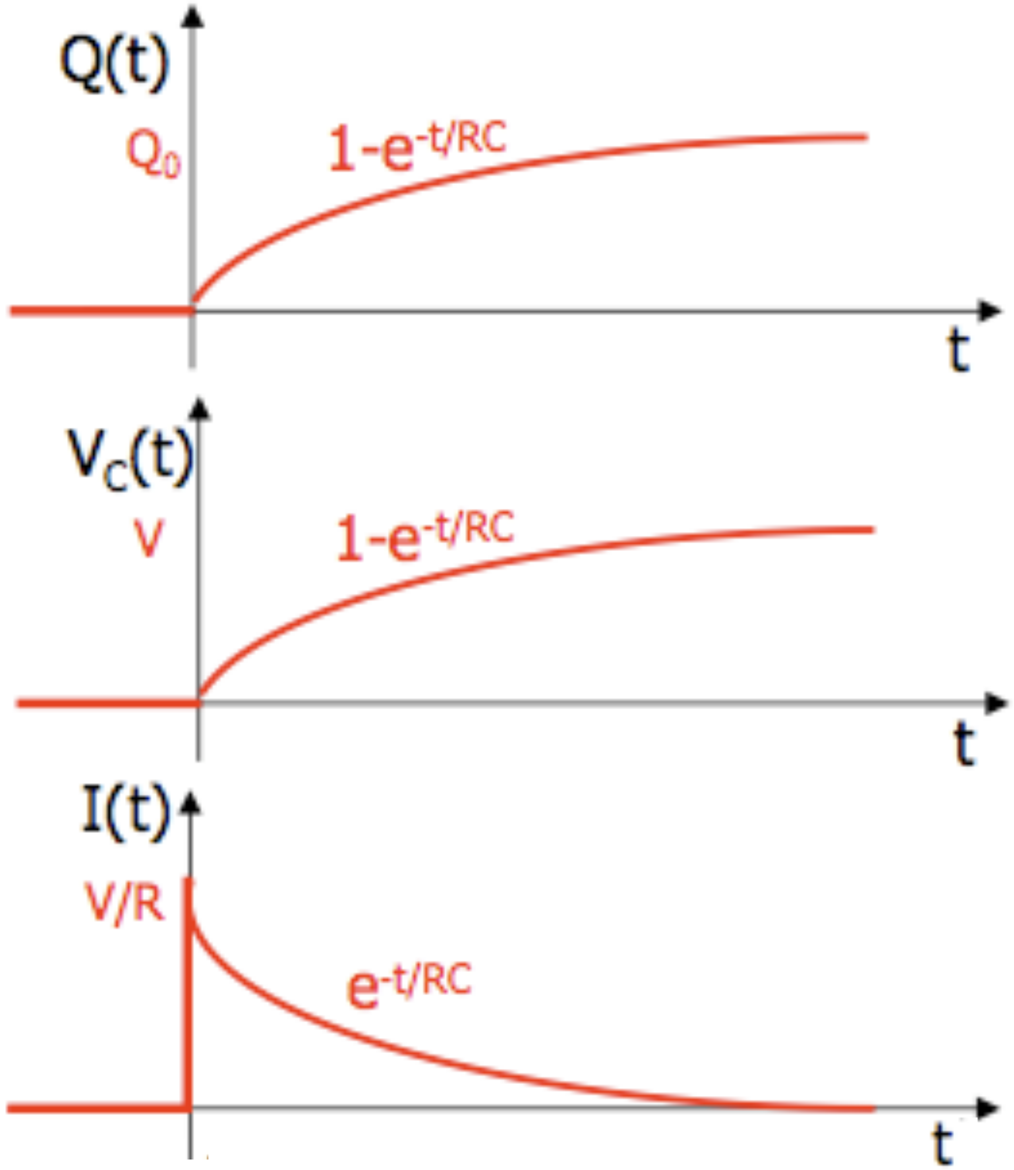
Integrating between t = 0 and t :

$$\int_{Q=0}^{Q=Q(t)} \frac{dQ'}{Q'} = -\frac{1}{RC}\int_{t=0}^{t=t} dt \Rightarrow \ln\frac{Q'(t)}{Q'(0)} = \ln\frac{Q(t) - CV}{-CV} = -\frac{t}{RC}$$

$$\Rightarrow \frac{Q(t) - CV}{-CV} = e^{-\frac{t}{RC}} \Rightarrow Q(t) = CV\left(1 - e^{-\frac{t}{RC}}\right)$$

Solution

Graphical solution



$$Q(t) = CV\left(1 - e^{-\frac{t}{RC}}\right)$$

$$V_C(t) = \frac{Q(t)}{C} = V\left(1 - e^{-\frac{t}{RC}}\right)$$

$$I(t) = \frac{dQ(t)}{dt} = \frac{V}{R} e^{-\frac{t}{RC}}$$

Important comments

Solution of RC circuit: $V_C(t) = \frac{Q(t)}{C} = V\left(1 - e^{-\frac{t}{RC}}\right)$ $\quad I(t) = \frac{dQ(t)}{dt} = \frac{V}{R} e^{-\frac{t}{RC}}$

Are Kirchhoff's laws valid at any moment in time?

$$V - \frac{Q}{C} - IR = V - V\left(1 - e^{-\frac{t}{RC}}\right) - R\frac{V}{R} e^{-\frac{t}{RC}} = 0 \qquad \text{OK!}$$

Asymptotic behavior of the capacitor:

At t = 0: I = V/R as if C were a short circuit

At t = infinity, I = 0 as if C were an open circuit

Conclusion: no need to solve the differential equation!

Solution is an exponential with time constant RC

Asymptotic behavior of C gives initial/final values for V(t) and I(t)

Time constant of RC circuit

Simple RC circuit with

V_{emf} = 3 V

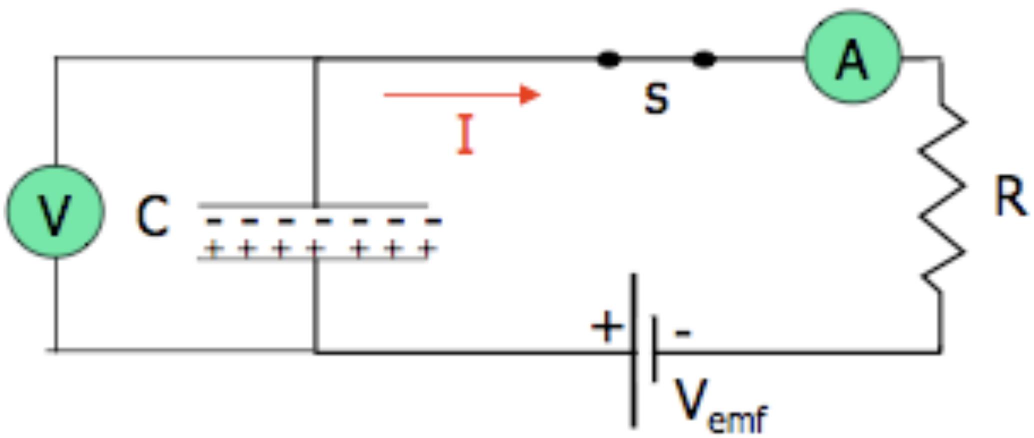
C = 1.3 F

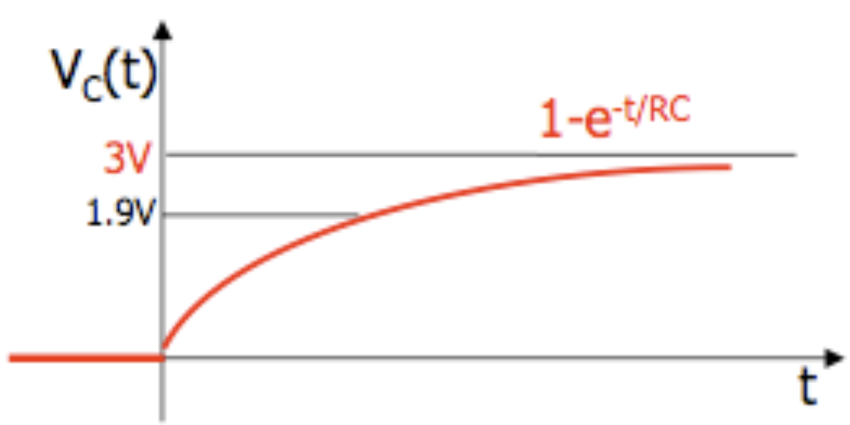
R = 11.7 Ω

Questions:

What are V_C and I?

Verify that time constant is RC: how long does it take to charge C ?





$$V_C(t) = V_{emf}\left(1 - e^{-\frac{t}{RC}}\right)$$

Note: R and C VERY large!

$$RC = 15.2\,s$$

If formula is correct →

V_C = V (1 - 1/e)=1.9 V when t=15.2

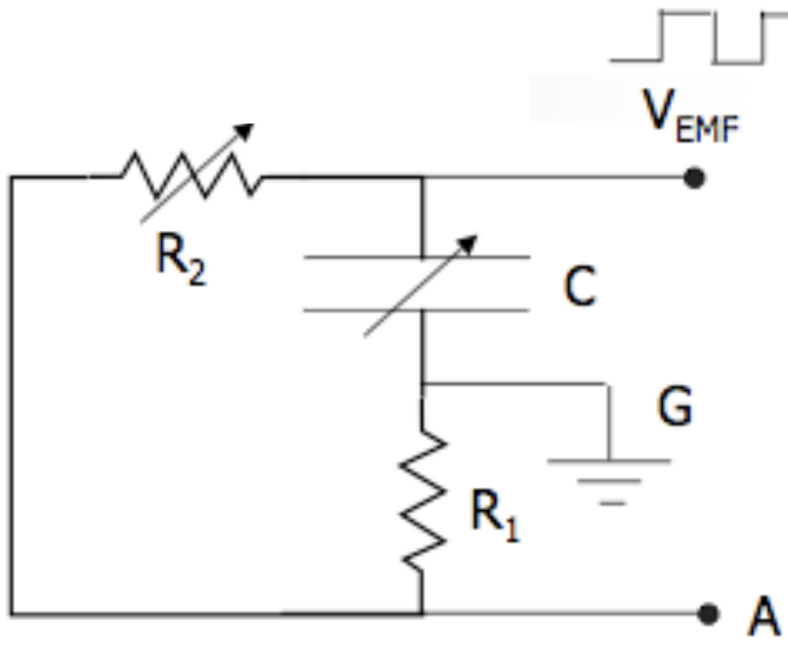
Verify time constant

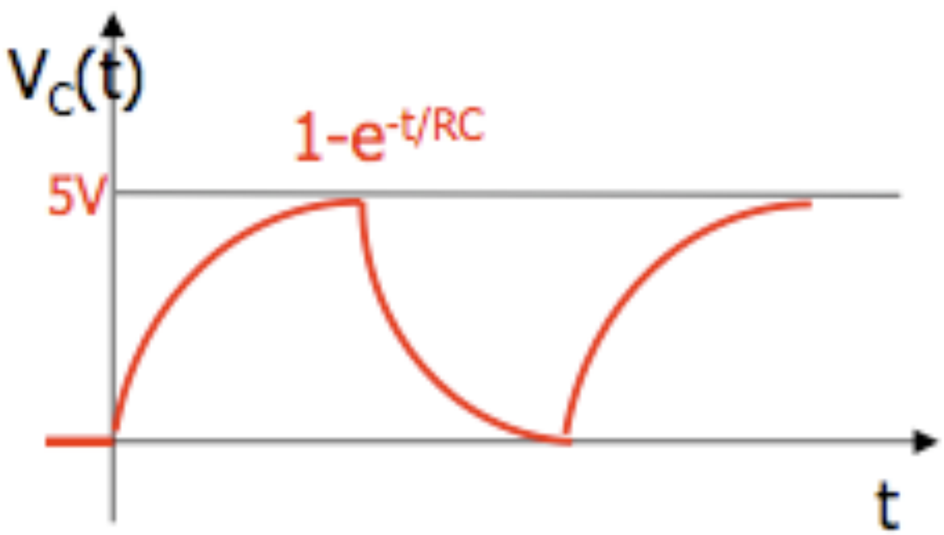
RC circuit with

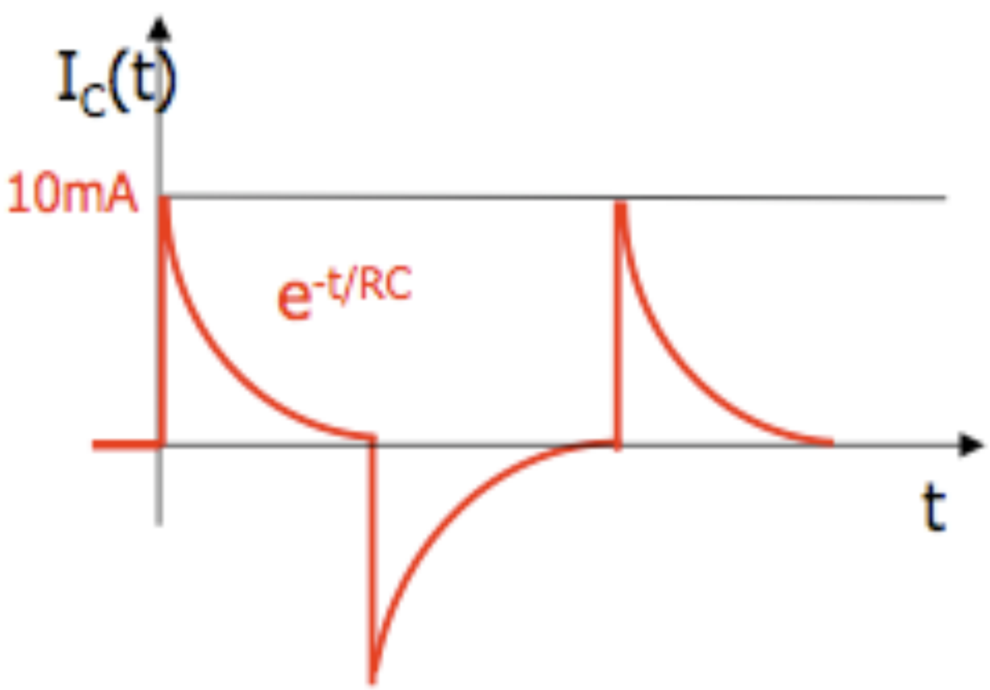
- V_{EMF}= squared 5 V pulses
- Variable C initially = 0.3 μF
- Variable R_2 initially = 400 Ω
- R_1= 100 Ω

Display on scope V_C and $I(R_1)$

- Verify RC=150 μs

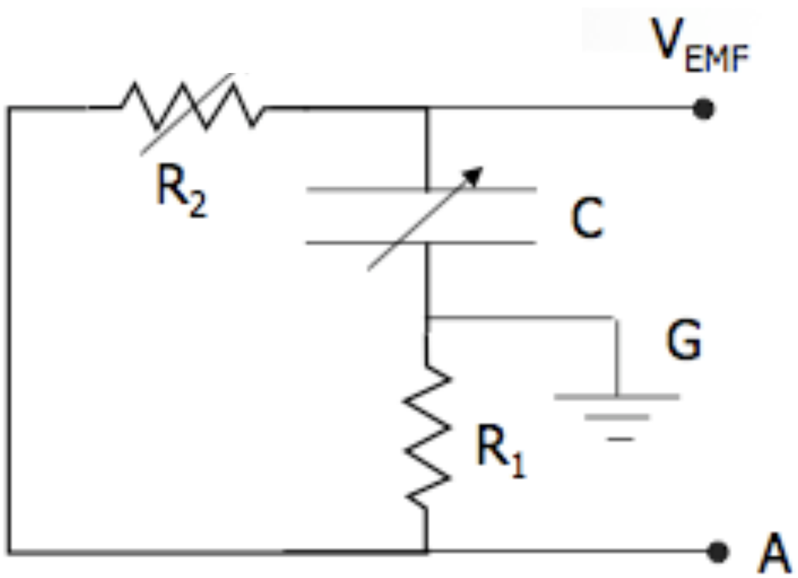






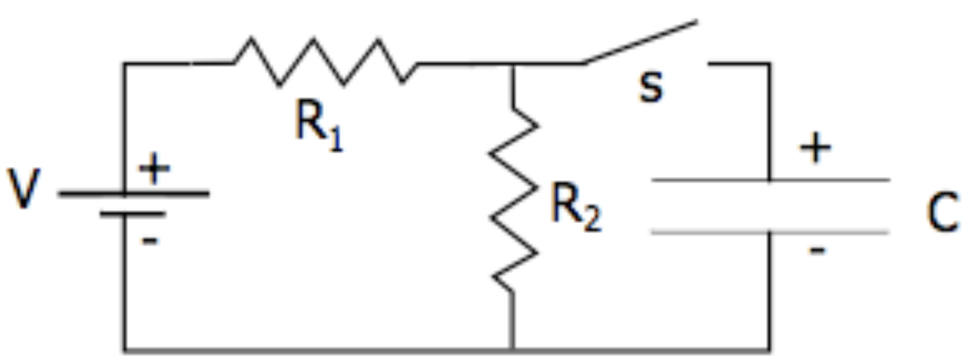
Verify time constant

- RC circuit with
  - V_{EMF}= squared 5 V pulses
  - Variable C initially = 0.3 μF
  - Variable R_2 initially = 400 Ω
  - R1= 100 Ω
- Let's now change the settings!
  - What happens when we double C?
    - $\tau_1 = RC' = 2RC = 2\tau_0 \rightarrow V\ (I_{AG})$ raises (falls) twice as fast
  - How should we change R_2 to have the same effect?
    - $R' = 2R = 2(R_1 + R_2') \rightarrow R_2'$: 400→900 Ω



More complicated RC circuits

- What if the RC circuit is more than just a series of R and C?
- Consider the following circuit:



Calculate Q(t) on the capacitor

Solution:

- Kirchoff's laws will solve it: TEDIOUS!
- Use Thevenin's Theorem!

Thevenin equivalence

Thevenin's theorem:

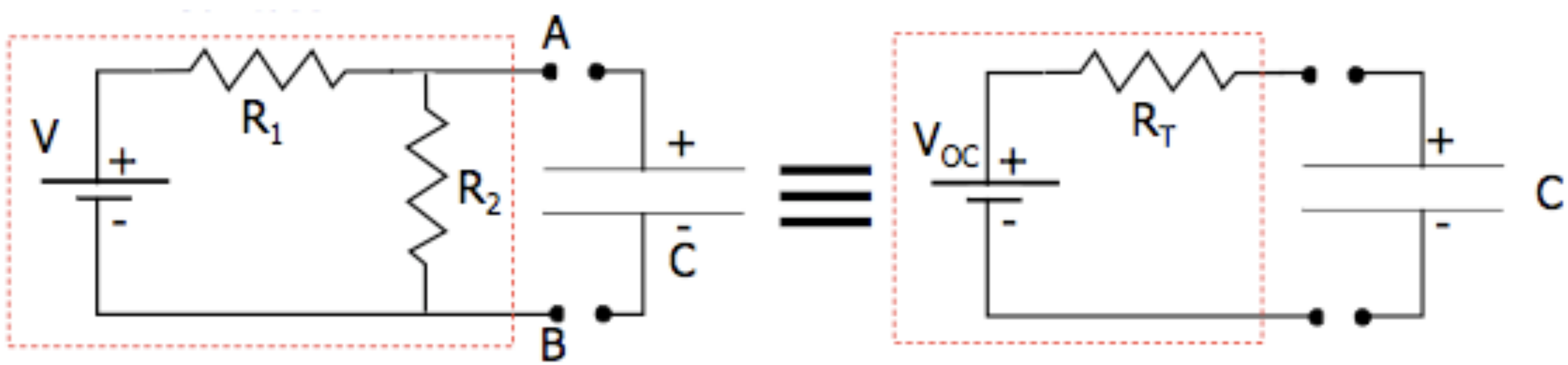
Any combination of resistors and EMFs with 2 terminals can be replaced with a series circuit of an emf V_{OC} and a resistor R_T where

V_{OC} is the open circuit voltage

$R_T = V_{OC}/I_{short}$ where I_{short} is the current going through the shorted terminals

or $R_T = R_{eq}$ with all the EMF shorted

In our case:



Once the circuit is reduced, the solution is known: $Q(t) = CV_{OC}\left(1 - e^{-\frac{t}{R_T C}}\right)$

Thevenin's demonstration

Prove that V_{OC} is the open circuit voltage

Since

$$Q(t) = CV_{OC}\left(1 - e^{-\frac{t}{R_T C}}\right) \Rightarrow V_C(t) = V_{OC}\left(1 - e^{-\frac{t}{R_T C}}\right)$$

So V_{OC} is the asymptotic V for the capacitor

Since for t → infinity, C → open circuit: V_{OC} = V of the open circuit

Prove that $R_T = V_{OC}/I_{short}$ with I_{short} = current through shorted terminals

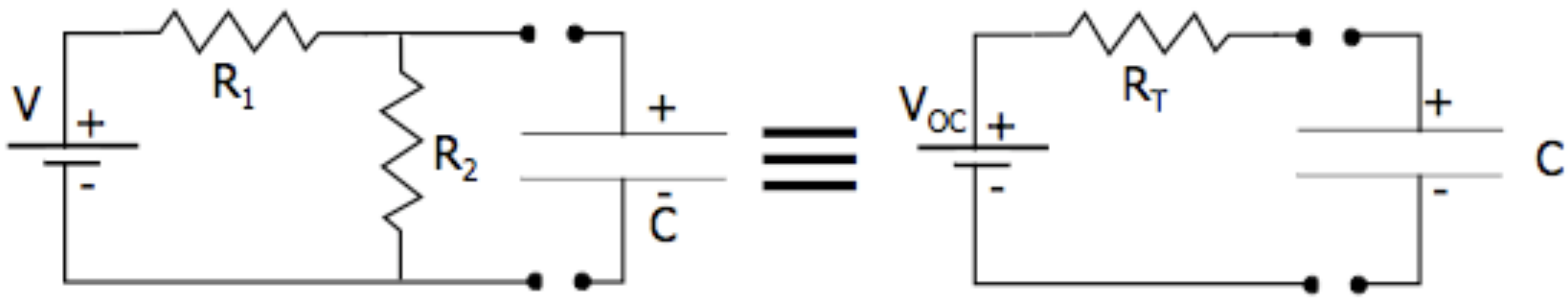
There is only one current going through the reduced circuit

At t = 0, C behaves like a short → At t = 0 $I_{short} = V_{OC}/R_T$

→ $R_T = V_{OC}/I_{short}$

Solve the actual problem

Calculate V_{OC} and $R_T = V_{OC}/I_{short}$ for our problem:



$$V_{OC} = \frac{V}{R_1 + R_2} R_2$$

Shorting C is makes R_2 irrelevant in the circuit:

$$I_{short} = \frac{V}{R_1}$$

$$\Rightarrow Q(t) = C \frac{VR_2}{R_1 + R_2}\left(1 - e^{-\frac{t(R_1+R_2)}{R_1 R_2 C}}\right)$$

$$R_{Thevenin} = \frac{V_{OC}}{I_{short}} = \frac{R_1 R_2}{R_1 + R_2}$$

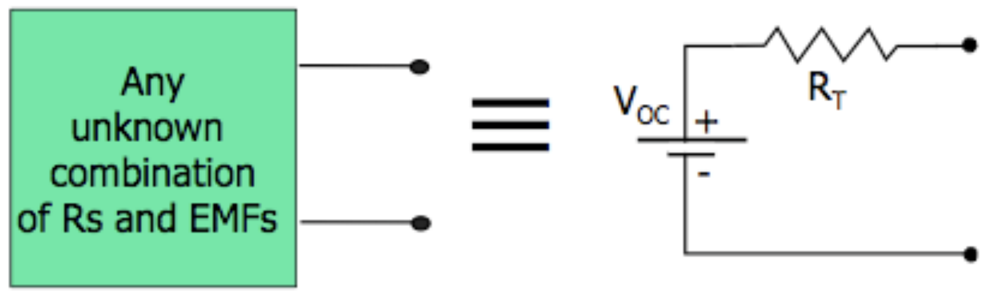
$$\Rightarrow I(t) = \frac{V}{R_1} e^{-\frac{t(R_1+R_2)}{R_1 R_2 C}}$$

Note : This is R1 || R2, same resistance we would get if we shorted EMF!

Thoughts on Thevenin

The importance of Thevenin:

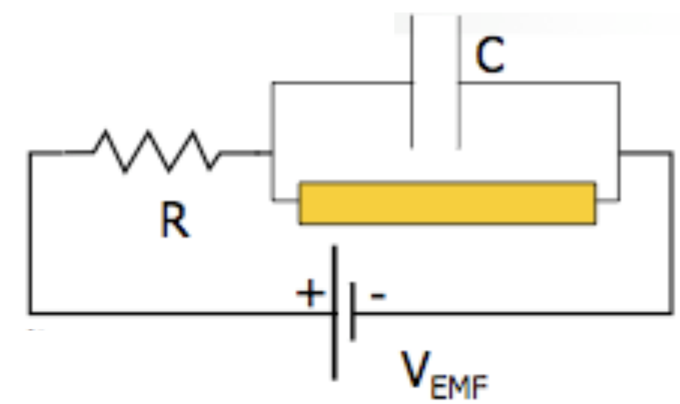
When we have a messy system or resistors and EMFs, we can reduce it to a simple R+EMF in series just measuring I_{short} and V_{open}:



Careful:

Thevenin works only when the elements in the box follow Ohm's law, i.e. linear relation between V and I

Oscillating circuit



RC circuit with:

- $V_{EMF} = 1$ kV
- $C = 0.1\ \mu F$
- $R = 2.5\ M\Omega$
- Fluorescent light in parallel with capacitor
- (R_{FL} <<< R when current flows; ~infinite otherwise)

Why is light flashing at f ~ 1Hz?

- Initially the capacitor will start charging (no current thru lamp)
- When V_C>certain value ~ 1kV current flows thru fluorescent light discharging the capacitor very quickly
- The process will start again
  - $f \sim 1/\tau = 1/RC = 4$ Hz

Note: charging and discharging time constants are very different!

Charging: fluorescent light is ~ open circuit: $\tau_{charge} = RC$

Discharge: fluorescent light has a (very small) resistance R_{FL}

Thevenin: $R_T = R \,||\, R_{FL} \sim R_{FL}$

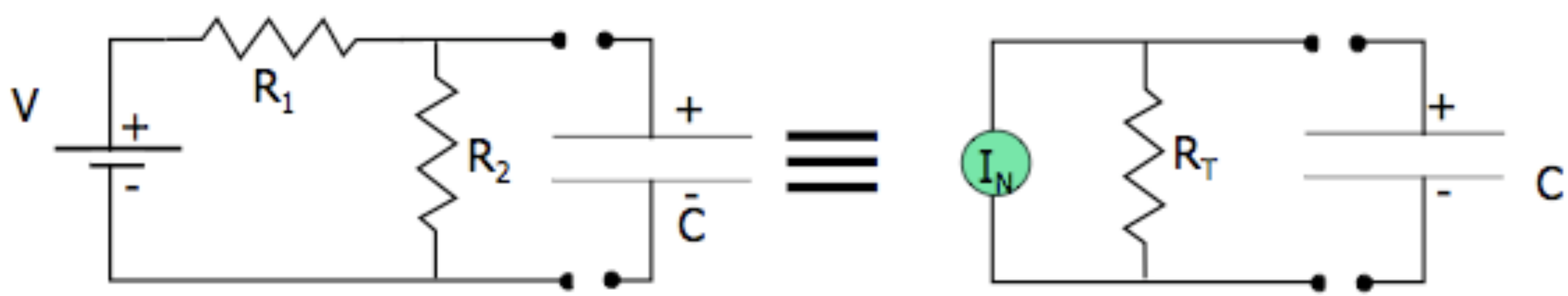
$\tau_{discharge} = R_T C \sim R_{FL} C << \tau_{charge}$

Norton's theorem

Any combination of resistors and EMFs with 2 terminals can be replaced with a parallel combination of a current generator I_N and a resistor R_T where

R_T is the equivalent resistance of the circuit with all the EMF shorted and all the current sources open (same as Thevenin!)

$I_N = V_{OC}/R_T$



$$R_T = R_1 \,||\, R_2 = \frac{R_1 R_2}{R_1 + R_2}$$

$$I_N = \frac{V_{OC}}{R_T} = \frac{V R_2}{R_1 + R_2} \frac{R_1 + R_2}{R_1 R_2} = \frac{V}{R_1}$$