Mathematical Methods Problems

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August 31, 2020

Assignment #1 - Review of Things You Should Know Already

1-Partial Fractions

Resolve the following into partial fractions in such a way that x does not appear in any numerator:

(a) $\frac{2x^2+x+1}{(x-1)^2(x+3)}$

(b) $\frac{x^2-2}{x^3+8x^2+16x}$

(c) $\frac{x^3-x-1}{(x+3)^3(x+1)}$

2-Binomial Expansion

Use a binomial expansion to evaluate $1/\sqrt{4.2}$ to five places of decimals, and compare it with the accurate answer obtained using a calculator.

3-An Integral

Express $x^2(ax+b)^{-1}$ as the sum of powers of x and another integrable term, and hence evaluate

$$\int_0^{b/a} \frac{x^2}{ax+b} dx$$

4-Integration by Parts

By integrating by parts twice, prove that I_n as defined in the first equality below for positive integers n has the value given in the second equality:

$$I_n = \int_0^{\pi/2} \sin n\theta \cos\theta d\theta = \frac{n - \sin(n\pi/2)}{n^2 - 1}$$

5-Some Complex Numbers

Evaluate

(a) $\mathrm{Re}(\exp 2iz)$

(b) $\mathrm{Im}(\cosh^2 z)$

(c) $(-1+\sqrt{3}i)^{1/2}$

(d) $\exp\left(i^{1/2}\right)$

(e) $\exp\left(i^3\right)$

(f) $\mathrm{Im}(2^{i+3})$

(g) i^i

(h) $\ln\left[(\sqrt{3}+i)^3\right]$

6-Hyperbolic Functions

Use the definitions and properties of hyperbolic functions to do the following:

(a) Solve $\cosh x = \sinh x + 2\,\mathrm{sech}\, x$

(b) Show that the real solution x of $\tanh x = \mathrm{cosech}\, x$ can be written in the form $x = \ln\left(u+\sqrt{u}\right)$. Find an explicit value for u.

(c) Evaluate $\tanh x$ when x is the real solution of $\cosh 2x = 2\cosh x$.

7-Maclaurin Series

Find the Maclaurin series for

(a) $\ln\left(\frac{1+x}{1-x}\right)$

(b) $(x^2+4)^{-1}$

(c) $\sin^2 x$

8-Limits

Evaluate the following limits:

(a) $\lim_{x\to 0} \frac{\sin 3x}{\sinh x}$

(b) $\lim_{x\to 0} \frac{\tan x - \tanh x}{\sinh x - x}$

(c) $\lim_{x\to 0} \frac{\tan x - x}{\cos x - 1}$

(d) $\lim_{x\to 0} \left(\frac{\csc x}{x^3} - \frac{\sinh x}{x^5}\right)$

9-Partial Derivatives in Thermodynamics

Show that

$$df = y(1 + x - x^2)dx + x(x + 1)dy$$

is not an exact differential.

Find the differential equation that a function $g(x)$ must satisfy if $d\phi = g(x)df$ is to be an exact differential. Verify that $g(x) = e^{-x}$ is a solution of this equation and deduce the form of $\phi(x, y)$.

10-Stationary Points

Locate the stationary points of the function

$$f(x, y) = (x^2 - 2y^2) \exp\left[-(x^2 + y^2)/a^2\right]$$

where a is a non-zero constant.

Sketch the function along the x- and y-axes and hence identify the nature and values of the stationary points.

11-Jacobians in Thermodynamics

The first law of thermodynamics can be expressed as

$$dU = TdS - PdV$$

By calculating and equating $\partial^2 U/\partial Y \partial X$ and $\partial^2 U/\partial X \partial Y$, where X and Y are an unspecified pair of variables (drawn from P,V,T and S), prove that

$$\frac{\partial(S, T)}{\partial(X, Y)} = \frac{\partial(V, P)}{\partial(X, Y)}$$

Using the properties of Jacobians, deduce that

$$\frac{\partial(S, T)}{\partial(V, P)} = 1$$

12-Vector Identity (use ε_{ijk})

Prove, by writing it out in component form, that

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$$

and deduce the result that the operation of forming the vector product is non-associative, i.e.,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b}) \times \mathbf{c})$$

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Assignment #2 - More Review and Fourier Series

Some More review Problems:

1-Pauli Matrices

The four matrices $\mathbf{S}_x$, $\mathbf{S}_y$, $\mathbf{S}_z$ and I are defined by

$$\mathbf{S_x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \mathbf{S}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \mathbf{S}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where $i^2 = -1$. Show that $\mathbf{S}_x^2$ and $\mathbf{S}_x\mathbf{S}_y = i\mathbf{S}_z$, and obtain similar results by permutting x, y and z. Given that $\mathbf{v}$ is a vector with Cartesian components (v_x, v_y, v_z), the matrix $\mathbf{S}(\mathbf{v})$ is defined as

$$\mathbf{S}(\mathbf{v}) = v_x\mathbf{S}_x + v_y\mathbf{S}_y + v_z\mathbf{S}_z$$

Prove that, for general non-zero vectors $\mathbf{a}$ and $\mathbf{b}$,

$$\mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{b}) = \mathbf{a}\cdot\mathbf{b} + i\mathbf{S}(\mathbf{a}\times\mathbf{b})$$

Without further calculation, deduce that $\mathbf{S}(\mathbf{a})$ and $\mathbf{S}(\mathbf{b})$ commute if and only if $\mathbf{a}$ and $\mathbf{b}$ are parallel vectors.

2-Gram-Schmidt

Using the Gram?Schmidt procedure:

(a) construct an orthonormal set of vectors from the following:

$$\mathbf{x}_1 = (0\ 0\ 1\ 1)^T \quad , \quad \mathbf{x}_2 = (1\ 0\ -1\ 0)^T$$

$$\mathbf{x}_3 = (1\ 2\ 0\ 2)^T \quad , \quad \mathbf{x}_4 = (2\ 1\ 1\ 1)^T$$

(b) find an orthonormal basis, within a four-dimensional Euclidean space, for the subspace spanned by the three vectors

$$(1\ 2\ 0\ 0)^T\ ,\ (3\ -1\ 2\ 0)^T\ ,\ (0\ 0\ 2\ 1)^T$$

3-Eigenvectors

By finding the eigenvectors of the Hermitian matrix

$$H = \begin{pmatrix} 10 & 3i \\ -3i & 2 \end{pmatrix}$$

construct a unitary matrix U such that $U^\dagger H U = \Lambda$, where Λ is a real diagonal matrix.

4-Simultaneous Equations

Solve the following simultaneous equations for x_1, x_2 and x_3, using matrix methods:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 3x_1 + 4x_2 + 5x_3 &= 2 \\ x_1 + 3x_2 + 4x_3 &= 3 \end{aligned}$$

5-Vector Identity (use ε_{ijk})

Verify by direct calculation that

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$$

6-Vector Field

The vector field $\mathbf{F}$ is defined by

$$\mathbf{F} = 2xz\mathbf{i} + 2yz^2\mathbf{j} + (x^2 + 2y^2z - 1)\mathbf{k}$$

Calculate $\nabla \times \mathbf{F}$ and deduce that $\mathbf{F}$ can be written $\mathbf{F} = \nabla\phi$. Determine the form of ϕ.

7-Fourier Series Problems

(1) Find the Fourier series of the function $f(x) = x$ in the range $-\pi < x \leq \pi$. Hence show that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

(2) Find the Fourier coefficients in the expansion of $f(x) = \exp(x)$ over the range $-1 < x < 1$. What value will the expansion have when x = 2?

(3) Show that the Fourier series for the function $y(x) = |x|$ in the range $-\pi \leq x < \pi$ is

$$y(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\cos(2m+1)x}{(2m+2)2}$$

By integrating this equation term by term from 0 to x, find the function $g(x)$ whose Fourier series is

$$\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\sin(2m+1)x}{(2m+2)3}$$

Deduce the value of the sum S of the series

$$1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots$$

(4) By finding a cosine Fourier series of period 2 for the function f(t) that takes the form $f(t) = \cosh(t-1)$ in the range $0 \leq t \leq 1$, prove that

$$\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2+1} = \frac{1}{e^2-1}$$

Deduce values for the sums $\sum(n^2\pi^2+1)^{-1}$ over odd n and even n separately.

(5) Express the function $f(x) = x^2$ as a Fourier sine series in the range $0 < x \leq 2$ and show that it converges to zero at $x = \pm 2$.

(6) Demonstrate explicitly for the odd (about $x = 0$) square-wave function that Parseval?s theorem is valid. You will need to use the relationship

$$\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2} = \frac{\pi^2}{8}$$

Show that a filter that transmits frequencies only up to $8\pi/T$ will still transmit more than 90% of the power in a square-wave voltage signal of period TT.

(7) Find the complex Fourier series for the periodic function of period 2π defined in the range $-\pi \leq x \leq \pi$ by $y(x) = \cosh(x)$. By setting $x = 0$ prove that

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1} = \frac{1}{2}\left(\frac{\pi}{\sinh\pi} - 1\right)$$

Assignment #3 - Fourier and Laplace Transforms

1-Fourier Transform

Find the Fourier transform of the function $f(t) = \exp(-|t|)$.

(a) By applying Fourier?s inversion theorem prove that

$$\frac{\pi}{2}\exp(-|t|) = \int_0^\infty \frac{\cos\omega t}{1+\omega^2}d\omega$$

(b) By making the substitution $\omega\tan\theta$, demonstrate the validity of Parseval?s theorem for this function.

2-Properties of Fourier Transform

Use the general definition and properties of Fourier transforms to show the following.

(a) If $f(x)$ is periodic with period a then $\tilde{f}(k) = 0$ unless $ka = 2\pi n$ for integer n.

(b) The Fourier transform of $tf(t)$ is $id\tilde{f}(\omega)/d\omega$.

(c) The Fourier transform of $f(mt+c)$ is

$$\frac{e^{i\omega c}/m}{m}\tilde{f}\left(\frac{\omega}{m}\right)$$

3-Fourier Transform

Find the Fourier transform of $H(x-a)e^{-bx}$, where $H(x)$ is the Heaviside function.

4-Fourier Transform

Find the Fourier transform of the unit rectangular distribution

$$f(t) = \begin{cases} 1 & |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

Determine the convolution of f with itself and, without further integration, deduce its transform. Deduce that

$$\int_{-\infty}^{\infty} \frac{\sin^2 \omega}{\omega^2} d\omega = \pi$$

$$\int_{-\infty}^{\infty} \frac{\sin^4 \omega}{\omega^4} d\omega = \frac{2\pi}{3}$$

5-Fourier Transform

Find the Fourier transform specified in part (a) and then use it to answer part (b).

(a) Find the Fourier transform of

$$f(\gamma, p, t) = \begin{cases} e^{-\gamma t} \sin pt & t > 0 \\ 0 & t < 0 \end{cases}$$

where $\gamma (> 0)$ and p are constant parameters.

(b) The current $I(t)$ flowing through a certain system is related to the applied voltage $V(t)$ by the equation

$$I(t) = \int_{-\infty}^{\infty} K(t-u) V(u) du$$

where

$$K(\tau) = a_1 f(\gamma_1, p_1, \tau) + a_2 f(\gamma_2, p_2, \tau)$$

The function $f(\gamma, p, t)$ is given in part (a) and all the a_1, γ_i (> 0) and p_i are fixed parameters. By considering the Fourier transform of $I(t)$, find the relationship that must hold between a_1 and a_2 if the total net charge Q passed through the system (over a very long time) is to be zero for an arbitrary applied voltage.

6-Using Fourier Transform

Prove the equality

$$\int_0^{\infty} e^{-2at} \sin^2 at dt = \frac{1}{\pi} \int_0^{\infty} \frac{a^2}{4a^4 + \omega^4} d\omega$$

7-Auto-Correlation

Calculate directly the auto-correlation function $a(z)$ for the product $f(t)$ of the exponential decay distribution and the Heaviside step function,

$$f(t) = \frac{1}{\lambda} e^{-\lambda t} H(t)$$

Use the Fourier transform and energy spectrum of $f(t)$ to deduce that

$$\int_{-\infty}^{\infty} \frac{e^{i\omega z}}{\lambda^2 + \omega^2} d\omega = \frac{\pi}{\lambda} e^{-\lambda |z|}$$

8-Laplace Transform

Find the Laplace transforms of $t^{-1/2}$ and $t^{1/2}$, by setting $x^2 = ts$ in the result

$$\int_0^{\infty} \exp(-x^2) dx = \frac{1}{2}\sqrt{\pi}$$

9-Inverse Laplace Transform

Find the functions $y(t)$ whose Laplace transforms are the following:

(a) $1/(s^2 - s - 2)$

(b) $2s/[(s+1)(s^2+4)]$

(c) $e^{-(\gamma+s)t_0}/[(s+\gamma)^2 + b^2]$

10-Properties of Laplace Transform

Use the properties of Laplace transforms to prove the following without evaluating any Laplace integrals explicitly:

(a) $\mathcal{L}[t^{5/2}] = \frac{15}{8}\sqrt{\pi} s^{-7/2}$

(b) $\mathcal{L}[(\sinh at)/t] = \frac{1}{2} \ln [(s+a)/(s-a)] \quad , \quad s > |a|$

(c) $\mathcal{L}[\sinh at \cos bt] = a(s^2 - a^2 + b^2)[(s-a)^2 + b^2]^{-1}[(s+a)^2 + b^2]^{-1}$

11-Using Laplace Transform

Show that the Laplace transform of $f(t-a)H(t-a)$, where $a \geq 0$, is $e^{-as}\tilde{f}(s)$, and that, if $g(t)$ is a periodic function of period T, $\tilde{g}(s)$ can be written as

$$\frac{1}{1-e^{-sT}}\int_0 Te^{-st}g(t)dt$$

(a) Sketch the periodic function defined in $0 \leq t \leq T$ by

$$g(t) = \begin{cases} 2t/T & 0 \leq t < T/2 \\ 2(1-t/T) & T/2 \leq t \leq T \end{cases}$$

and, using the previous result, find its Laplace transform.

(b) Show, by sketching it, that

$$\frac{2}{T}[f(t-a)H(t-a)] = \int_0^\infty f(t-a)H(t-a)e^{-st}dt$$

is another representation of g(t) and hence derive the relationship

$$\tanh x = 1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-2nx}$$

Assignment #4-Ordinary Differential Equations (ODEs) - Part 1

1-Separation of Variables

Solve the following equations by separation of the variables:

(a) $y' - xy^3 = 0$

(b) $y' \tan^{-1} x - y(1 + x^2)^{-1} = 0$

(c) $x^2y' + xy^2 = 4y^2$

2-Exact Equations

Show that the following equations either are exact or can be made exact, and solve them:

(a) $y(2x^2y^2 + 1)y' + x(y^4 + 1) = 0$

(b) $2xy' + 3x + y = 0$

(c) $(\cos^2 x + y \sin 2x)y' + y^2 = 0$

3-Integrating Factor

By finding an appropriate integrating factor, solve

$$\frac{dy}{dx} = -\frac{2x^2 + y^2 + x}{xy}$$

4-An ODE

Solve

$$(y - x)\frac{dy}{dx} + 2x + 3y = 0$$

5-ODE and Laplace Transform

One of the properties of Laplace transforms is that the transform of the n^{the} derivative of a function $f(t)$ is given by

$$\mathcal{L}\left[\frac{d^n f}{dt^n}\right] = s^n \tilde{f} - s^{n-1} f(0) - s^{n-2}\frac{df}{dt}(0) - \cdots - \frac{df^{n-1}}{dt^{n-1}}(0)\ ,\ \text{for } s > 0$$

Using this and the result about the Laplace transform of $tf(t)$

$$\mathcal{L}[[t^n f(t)] = (-1)^n \frac{d^n \tilde{f}(s)}{ds^n} \qquad \text{for } n = 1, 2, 3, ...$$

show, for a function $y(t)$ that satisfies

$$t\frac{dy}{dt} + (t-1)y = 0 \qquad (*)$$

with $y(0)$ finite, that $\tilde{y}(s) = C(1+s)^{-2}$ for some constant C.

Given that

$$y(t) = t + \sum_{n=2}^{\infty} a_n t^n$$

determine C and show that $a_n = (?-1)^{n-1}/(n-1)!$. Compare this result with that obtained by integrating (*) directly.

6-An ODE

Solve

$$x(1-2x^2y)\frac{dy}{dx} + y = 3x^2y^2$$

given that $y(1) = 1/2$.

7-Some 1st-Order ODEs

Solve the following first-order equations for the boundary conditions given:

(a) $y' - (y/x) = 1 \quad , \quad y(1) = -1$

(b) $y' - y\tan x = 1 \quad , \quad y(\pi/4) = 3$

(c) $y' - y^2/x^2 = 1/4 \quad , \quad y(1) = 1$

(d) $y' - y^2/x^2 = 1/4 \quad , \quad y(1) = 1/2$

8-Coupled ODEs

An electronic system has two inputs, to each of which a constant unit signal is applied, but starting at different times. The equations governing the system thus take the form

$$\dot{x} + 2y = H(t)$$
$$\dot{y} - 2x = H(t-3)$$

Initially (at $t = 0$), $x = 1$ and $y = 0$; find $x(t)$ at later times.

9-An ODE

Solve the differential equation

$$\sin x \frac{dy}{dx} + 2y \cos x = 1$$

subject to the boundary condition $y(\pi/2) = 1$.

10-An ODE

Find the solution of

$$(2 \sin y - x)\frac{dy}{dx} = \tan y$$

if (a) $y(0) = 0$, and (b) $y(0) = \pi/2$.

11-Integrating Factor

Find the general solution of

$$\frac{dy}{dx} - y = \cos x$$

Use integrating factor theory.

Assignment #5-Ordinary Differential Equations (ODEs) - Part 2

1-An ODE

Solve the differential equation

$$\frac{d^2 f}{dt^2} + 6\frac{df}{dt} + 9f = e^{-t}$$

subject to the conditions $f = 0$ and $df/dt = \lambda$ at $t = 0$.

2-An ODE

The function $f(t)$ satisfies the differential equation

$$\frac{d^2 f}{dt^2} + 8\frac{df}{dt} + 12f = 12e^{-4t}$$

subject to the conditions $f(0) = 0,\ f'(0) = 0,\ f(\ln\sqrt{2}) = 0$

3-Two ODEs

Find the general solutions of

(a) $\frac{d^3y}{dx^3} - 12\frac{dy}{dx} + 16y = 32x - 8$

(b) $\frac{d}{dx}\left(\frac{1}{y}\frac{dy}{dx}\right) + (2a\coth 2ax)\left(\frac{1}{y}\frac{dy}{dx}\right) = 2a^2$

where a is a constant.

4-ODEs and Laplace Transform

Use the method of Laplace transforms to solve

(a) $\frac{d^2f}{dt^2} + 5\frac{df}{dt} + 6f = 0 \quad , \quad f(0) = 1, f'(0) = -4$

(b) $\frac{d^2f}{dt^2} + 2\frac{df}{dt} + 5f = 0 \quad , \quad f(0) = 1, f'(0) = -0$

5-Variation of Parameters

Use the method of variation of parameters to find the general solutions of

(a) $\frac{d^2y}{dx^2} - y = x^n$

(b) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 2xe^x$

6-Wronskian gives second solution

Solve

$$2y\frac{d^3y}{dx^3} + 2\left(y + 3\frac{dy}{dx}\right)\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = \sin x$$

7-Wronskian gives second solution

For the ODE

$$x^2\frac{d^2y}{dx^2} + x(1-x)\frac{dy}{dx} + (1-x)y = 0$$

one solution is $y_1(x) = x$. Find the Wronskian and use it to find asecond solution.

8-Finding a clever trick

For the ODE with constant coefficients

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 5y = x^2 + 2\sin x$$

find the homogeneous solutions. There is a clever way to find the particular solution using only the non-homogeneous differential equation and the standard form of a power series with the coefficients written in terms of derivatives. See if you can figure it out. If not, then you have to use variation of parameters method to find the particular solution and this is extraordinarily difficult!!

9-Using Laplace transform

Using Laplace transforms determine the solution for the initial value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2t \quad , \quad y(0) = 0, y'(0) = 0$$

10-An ODE

Solve the ODE

$$u'' + 9u = x^2 + \sin 2x$$

11-An ODE

Solve the ODE

$$u'' + 6u' + 10u = \cos 2x \quad , \quad u(0) = u'(0) = 0$$

12-An ODE

Solve the ODE

$$\ddot{y} + 2\dot{y} + 10y = 26\sin 2t \quad y(0) = 1, \dot{y}(0) = 0$$

13-An ODE

Solve the ODE

$$x^2u'' + xu' - u = 9$$

14-An ODE

Solve the ODE

$$x^2u'' - 2xu' - 4u = x\cos x$$

Assignment #6-Ordinary Differential Equations (ODEs) - Part 3

1-Series Solution

Find two power series solutions about $z = 0$ of the differential equation

$$(1 - z^2)y'' - 3zy' + \lambda y = 0$$

Deduce that the value of λ for which the corresponding power series becomes an Nth-degree polynomial $U_N(z)$ is $N(N + 2)$. Construct $U_2(z)$ and $U_3(z)$.

2-Series Solution

Find solutions, as power series in z, of the equation

$$4zy'' + 2(1 - z)y' - y = 0$$

Identify one of the solutions and verify it by direct substitution.

3-Frobenius method

For the ODE

$$36x^2\frac{d^2y}{dx^2} + (5 - 9x^2)y = 0$$

use the method of Frobenius to find both homogeneous solutions.

4-Frobenius method

For the ODE

$$x^2\frac{d^2y}{dx^2} - 6y = 0$$

use the method of Frobenius to find both homogeneous solutions.

5-Frobenius method

Change the independent variable in the equation

$$f'' + 2(z - \alpha)f' + 4f = 0$$

from z to $x = z - \alpha$, and find two independent series solutions, expanded about $x = 0$, of the resulting equation. Deduce that the general solution of the original equation is

$$f(z,\alpha) = A(z-\alpha)e^{-(z-\alpha)^2} + B\sum_{m=0}^{\infty}\frac{(-4)^m m!}{(2m)!}(z-\alpha)^{2m}$$

with A and B arbitrary constants.

6-Frobenius method

Show that the equation

$$z^2y'' - \frac{3}{2}zy' + (1+z)y = 0$$

has an indicial equation with roots 2 and 1/2. Then show that the general solution is

$$\begin{aligned} y(z) = & 6a_0 z^2 \sum_{n=0}^{\infty}\frac{(-1)^n(n+1)2^{2n}}{(2n+3)!}z^n \\ & + b_0\left(z^{1/2} + 2z^{3/2} - \frac{z^{1/2}}{4}\sum_{n=2}^{\infty}\frac{(-1)^n 2^{2n}}{n(n-1)(2n-3)!}z^n\right) \end{aligned}$$

7-Power-series method - non-singular equations

Solve the ODE for a general solution using the power-series method (non-singular equations).

(a) $(1-x)u'' - u = 0$

(b) $(x^2-1)u'' - 4u = 0$

(c) $(4-x^2)u'' + 2u = 0 \quad , \quad u(0) = 0, u'(0) = 1$

(d) $u'' - x^2u' + u\sin x = 0 \quad , \quad u(0) = 0, u'(0) = 1$

8-Power-series method - singular equations

Solve the ODE for a general solution using the power-series method (singular equations).

(a) $2x(1-x)u'' + u' + u = 0$

(b) $4x^2(1-x)u'' - xu' + (1-x)u = 0$

(c) $2x^2u'' + x^2u' + (4/9)u = 0$

(d) $3xu'' + 2(1-x)u' - 4u = 0$

9-Wronskian Method

One solution of the ODE

$$x^2u'' - x(1+x)u' + u = 0$$

is $u_1(x) = xe^x$. Verify this. Determine a second solution.

10-Series + Wronskian Method

Solve

$$xu'' + (1-x)u' - u = 0$$

by finding the first solution using the series method and then finding the second solution using the Wronskian method.

11-Laguerre polynomials

By assuming a solution of the form $L_N(z) = \sum_{n=0}^{N} a_n z^n$ with $a_N \neq 0$, prove that the Laguerre equation

$$zy'' + (1-z)y' + \lambda y = 0$$

has polynomial solutions $L_N(z)$ if λ is a non-negative integer N, and determine the recurrence relationship for the polynomial coefficients. Hence show that an expression for $L_N(z)$, normalized in such a way that $L_N(0) = N!$, is

$$L_N(z) = \sum_{n=0} \frac{(-1)^n (N!)^2}{(N-m)!(N!)^2} z^n$$

Evaluate $L_3(z)$ explicitly.

12-Legendre's Equation

Determine the general solution near the origin for

(a) $(1 - x^2)u'' - 2xu' + 12u = 0$

(b) $(1 - x^2)u'' - 2xu' + 20u = 14x^2$ (need to find particular solution also)

(use solutions in text)

13-Bessel's Equation

Determine the general solution for

$$xu'' + u' + (x - 1/9x)u = 0$$

(use solutions in text)

14-Properties of Bessel Functions

Find an expression in terms of $J_1(x)$ and $J_0(x)$ for each integral below:

(a) $\int \frac{J_4(x)}{x} dx$

(b) $\int x^3 J_1(x) dx$

Assignment #7 - Partial Differential Equations (PDEs) - Part 1

Use the Separation of Variables Method for solution

Wave Equation

1-Vibrating string

A string of length L with fixed ends has a zero initial velocity and a displacement

$$y_0 = \begin{cases} 8hx/L & 0 < x < L/8 \\ 8h(L/4 - x)/L & L/8 < x < L/4 \\ 0 & L/4 < x < L \end{cases}$$

(this initial displacement might be caused by holding the string at the center and plucking half of it).

Find the displacement as a function of x and t.

Answer is:

$$y = \frac{16h}{\pi^2} \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi v t}{L}\right)$$

where

$$b_n = \frac{2\sin\left(\frac{n\pi}{8}\right) - \sin\left(\frac{n\pi}{4}\right)}{n^2}$$

2-Vibration of a Membrane

Separate the wave equation in 2-dimensional rectangular coordinates (x, y). Consider a rectangular membrane with corners $(0,0), (a,0), (a,b), (0,b)$ that is rigidly attached to supports along its sides. Show that its characteristic frequencies are

$$f_{nm} = \frac{v}{2}\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}$$

where (m, n) are positive integers. Sketch the normal modes of vibration corresponding to the first few frequencies, that is, indicate the nodal lines.

Now suppose that the membrane is square. Show that in this case there may be two or more normal modes corresponding to a single frequency. This is an example of a phenomenon called **degeneracy**.

3-Vibrating string

A string π meters long is started into motion by giving the middle one-half ($\pi/4 \le x \le 3\pi/4$) an initial velocity of 20 m/s (while keeping the initial displacement zero everywhere). The string is stretched until the wave speed is 60 m/s. Determine the resulting displacement of the string as a function of x and t. Answer is:

$$y(x,t) = \sum_{n=1}^{\infty} \frac{2}{3\pi n^2}\left(\cos\frac{n\pi}{4} - \cos\frac{3n\pi}{4}\right)\sin(60nt)\sin(nx)$$

Diffusion or Heat Flow Equation

1-Heat flow in a bar or slab

A bar 10 cm long with insulated sides is initially at 100°. Starting at $t = 0$, the ends are held at 0°. Find the temperature distribution in the bar at time t.

Answer is:

$$T = \frac{400}{\pi}\sum_{n,\text{odd}} \frac{1}{n} e^{-\left(\frac{n\pi\alpha}{10}\right)^2 t} \sin\left(\frac{n\pi x}{10}\right)$$

2-Heat flow in a bar or slab

In the initial steady-state distribution of an infinite slab of thickness$d$, the face $x = 0$ is at 0° and the face $x = d$ is at 100°. From $t = 0$ on, the $x = 0$ face is held at 100° and the $x = d$ face is held at 0°. Find the temperature distribution in the bar at time t.

Answer is:

$$T = 100 - \frac{100x}{d} - \frac{400}{\pi}\sum_{n,\text{even}} \frac{1}{n} e^{-\left(\frac{n\pi\alpha}{d}\right)^t} \sin\left(\frac{n\pi x}{d}\right)$$

You need to consider the $k = 0$ ($k_n = k_0 = 0$) solution.

3-Heat flow in a bar or slab

Two slabs, each 1 inch thick, each have one surface at 0° and the other at 100°. At $t = 0$, they are stacked with their 100∘ faces together and then the outside surfaces are held at 100∘. Find the temperature distribution for $t > 0$.

Answer is:

$$T = 100 + \frac{1}{\pi}\sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi\alpha}{2}\right)^2 t}\cos\left(\frac{n\pi x}{L}\right)$$

where

$$b_n = 400\begin{cases} 0 & \text{even } n \\ \frac{2}{n^2\pi^2} - \frac{1}{n\pi} & n = 1 + 4k \\ -\frac{2}{n^2\pi^2} - \frac{1}{n\pi} & n = 3 + 4k \end{cases}$$

You need to consider the $k = 0$ $(k_n = k_0 = 0)$ solution.

4-Heat flow in a bar or slab

A bar of length L with insulated sides has its ends also insulated from time $t = 0$ on. Initially the temperature distribution is $T = x$, where $x =$ distance from one end. Determine the temperature distribution inside the bar at time t. (Hint: you cannot neglect the $k = 0$ solutions in this case).

Answer is:

$$T = \frac{L}{2} - \frac{4L}{\pi^2}\sum_{n,\text{odd}} \frac{1}{n^2} e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}\sin\left(\frac{n\pi x}{10}\right)$$

5-Heat flow in a bar or slab

Two identical copper bars are each of length a. Initially, one is at 0∘ and the other is at 100∘. They are then joined together end to end and thermally isolated. Obtain in the form of a Fourier series and expression $T(x,t)$ for the temperature at any point a distance x from the join at a later time t.

Taking $a = 0.5$ m, estimate the time it takes for one of the free ends to reach a temperature of 55∘. The thermal conductivity of copper is 3.8×10^2 J $m^{-1}K^{-1}s^{-1}$ and its specific heat capacity is 3.4×10^6 $Jm^{-1}K^{-1}$.

Answer is:

$$T(x,t) = 50 + \frac{200}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin \frac{(2n+1)\pi x}{2a} e^{-\frac{D(2n+1)^2 \pi^2 t}{4a^2}}$$

Assignment #8 - Partial Differential Equations (PDEs) - Part 2

Use the Separation of Variables Method for solution

Diffusion or Heat Flow Equation

6-Heat flow in a bar or slab

Solve the Diffusion Equation for $T(x,t)$ under the following conditions. We have a laterally insulated 2 m long rod with conductivity 10^{-4} m^2/s and $T(x,0) = 100(2x - x^2)$, $T(0,t) = 0$, $T(2,t) = 0$.

Answer is:

$$T(x,t) = \sum_{n=1}^{\infty} A_n e^{-\pi^2 D n^2 t/4} \sin \frac{n\pi x}{2}$$

$$A_n = \int_0^2 100(2x - x^2) \sin \frac{n\pi x}{2} dx$$

Laplace Equation

Steady-state temperature in a rectangular plate

1-Semi-infinite plate

Find the steady-state temperature distribution for a semi-infinite plate if the temperature of the bottom edge is $T = f(x) = x$, the temperature of the sides is $0°$ and the width of the plate is 10 cm.

Answer is:

$$T = \frac{20}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-\frac{n\pi y}{10}} \sin\left(\frac{n\pi x}{10}\right)$$

2-Semi-infinite plate

Solve the semi-infinite plate problem if the bottom edge of width 20 cm is held at

$$T = \begin{cases} 0° & 0 < x < 10 \\ 100° & 10 < x < 20 \end{cases}$$

and the sides are at 0°.

Answer is:

$$T = \sum_n b_n e^{-(\frac{n\pi}{20})y} \sin\left(\frac{n\pi x}{20}\right)$$

$$b_n = \begin{cases} -\frac{400}{n\pi} & n \text{ even} \\ \frac{200}{n\pi} & n \text{ odd} \end{cases}$$

3-Finite plate

Solve problem 2 if the plate is cut off at height 10 cm and the temperature of the top edge is 0°.

Answer is:

$$T = \sum_n b_n \sinh\left(\frac{n\pi}{10}(10-y)\right) \sin\left(\frac{n\pi x}{10}\right)$$

$$b_n = \begin{cases} -\frac{400}{n\pi} & n \text{ even} \\ \frac{200}{n\pi} & n \text{ odd} \end{cases}$$

4-Finite plate

Find the temperature distribution in a rectangular plate 10 cm by 30 cm if two adjacent sides are held at 100° and the other two sides are at 0°.

Answer is:

$$T = \frac{400}{\pi} \sum_{n,\text{ odd}} \frac{1}{n} \left[\frac{1}{\sinh(3n\pi)} \sin\left(\frac{n\pi}{10}(30-y)\right) \sin\left(\frac{n\pi x}{10}\right) + \frac{1}{\sinh(n\pi/3)} \sin\left(\frac{n\pi}{10}(10-x)\right) \sin\left(\frac{n\pi y}{30}\right) \right]$$

5-Finite slab

Using the diffusion equation find the steady-state temperature distribution in a 1 m x 1 m slab if the flat surfaces are insulated and the edge conditions are as follows:

$$T(0,y) = 0 \ , \ \frac{\partial T}{\partial y}(x,0) = 0 \ , \ \frac{\partial T}{\partial y}(1,y) = 0 \ , \ T(x,1) = 100$$

Answer is:

$$T(x,1) = 100 = \sum_{n=1}^{\infty} A_n \sin \frac{2n-1}{2} \pi x (e^{\frac{2n-1}{2}\pi} + e^{-\frac{2n-1}{2}\pi})$$

$$A_n = frac400(2n-1)\pi(e^{\frac{2n-1}{2}\pi} + e^{-\frac{2n-1}{2}\pi})$$

Laplace Equation

Steady-state temperature in a cylinder

1-Semi-infinite cylinder

Find the steady-state temperature distribution in solid semi-infinite cylinder if the boundary temperatures are $u = 0$ at $r = 1$ and $u = y = r \sin\theta$ at $z = 0$.

Answer is:

$$T = \sum_{m=1}^{\infty} \frac{2}{k_m J_2(k_m)} J_1(k_m r) e^{-k_m t} \sin\theta \quad , \quad k_m = \text{ zeroes of } J_1$$

2-Water pipe

Water at 100° is flowing through a long pipe of radius $r = 1$ rapidly enough so that we may assume that the temperature is 100° at all points. At $t = 0$, the water is turned off and the surface of the pipe is maintained at 40° from then on (neglect wall thickness). Find the temperature distribution of the water as a function of r and t. (Hint: you only need to consider a cross-section of the pipe).

Answer is:

$$T = 40 + \sum_{m=1}^{\infty} \frac{120}{k_m J_1(k_m)} J_0(k_m r) e^{-(\alpha k_m)^2 t} \sin\theta \quad , \quad k_m = \text{ zeroes of } J_0$$

Assignment #9 - Partial Differential Equations (PDEs) - Part 3

Use the Separation of Variables Method for solution

Laplace Equation

Steady-state temperature in a cylinder

3-Circular plate

Separate Laplace's equation in two dimensions in polar coordinates and solve the r and θ equations. Remember that for the θ equation, only periodic solutions are of interest. Use your results to solve the problem of the steady state temperature in a circular plate (radius = a) if the upper semi-circular boundary is held at 100° and the lower is held at 0°.

Answer is:

$$T = 50 + \frac{200}{\pi} \sum_{n,\ \text{odd}} \left(\frac{r}{a}\right)^n \frac{\sin n\theta}{n} \qquad a = \text{ disk radius}$$

4-Circular annulus

Find the steady-state temperature distribution in a circular annulus of inner radius $r = 1$ and outer radius $r = 2$ if the inner circle is held at 0° and the outer circle has half of its circumference at 0° and half at 100°.(Hint: you cannot neglect r solutions corresponding to $k = 0$).

Answer is:

$$T = 50\frac{\log(r)}{\log(2)} + \frac{200}{\pi} \sum_{n,\ \text{odd}} \frac{1}{n}\left(\frac{r^n - r^{-n}}{2^n - 2^{-n}}\right)\sin n\theta \qquad \log \ = \ \text{natural logarithm}$$

5-Finite cylinder

A right circular cylinder is 1 m long and 2 m in diameter. Its left end and lateral surface are maintained at a temperature of 0° and its right end at 100°. Find the temperature at any interior point. Calculate the first three coefficients in the series expansion.

Answer is:

$T(r,z) = 29.4J_0(2.4r)\sinh(2.4z) - 0.86J_0(5.52r)\sinh(5.52z) + 0.03J_0(8.65r)\sinh(8.65z) - \cdots$

Laplace Equation Steady-state temperature in a sphere

1-Sphere

Find the steady-state temperature distribution inside a sphere of radius $r = 1$ when the surface temperatures are given by:

(a) $35(\cos\theta)^4$

(b) $\begin{cases} \cos\theta & 0 < \theta < \pi/2 \\ 0 & \pi/2 < \theta < \pi \end{cases}$

Answers are:

(a) $T(r,\theta) = 8r^4 P_4(\cos\theta) + 20r^2 P_2(\cos\theta) + 7P_0(\cos\theta)$

(b) $T(r,\theta) = \frac{1}{4}P_0(\cos\theta) + \frac{1}{2}rP_1(\cos\theta) + \frac{5}{16}r^2 P_2(\cos\theta) + \cdots$

2-Frozen sphere

A sphere (radius = a) initially at 0° has its surface kept at 100° from $t = 0$ on (for example, a frozen potato is tossed into in boiling water). Find the time-dependent temperature distribution. (Hint: subtract 100 from all temperatures, solve the problem and then add 100 to the solution; can you justify this procedure?). Show that the Legendre function required for this problem is P_0 and the r solution is

$$\frac{1}{\sqrt{r}} J_{1/2} \to j_0$$

Answers is:

$$T = 100 + \frac{200a}{\pi r} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\left(\frac{n\pi r}{a}\right) e^{-(n\pi\alpha/a)^2 t}$$

Laplace Equation

Electric Potentials

1-Electric Potentials Shells

Find the potential between two concentric spheres if the outer sphere is maintained at $V = 100$ and the potential on the inner sphere is maintained at zero. The radii are 2 m and 1 m, respectively.

Answers is:

$$V(r,\theta) = 200\left[1 - \frac{1}{r}\right]$$

2-Conducting sphere

A conducting sphere of radius a is cut around its equator and the two haves are connected to voltages of $+V$ and $-V$. Show that an expression for the potential at a point (r,θ,ϕ) anywhere inside the two hemispheres is

Answers is:

$$u(r,\theta,\phi) = V\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!(4n+3)}{2^{2n+1}n!(N+1)!}\left(\frac{r}{a}\right)^{2n+1}P_{2n+1}(\cos\theta)$$

You only need to modify some of the example in the text.

Assignment #10-Complex Variables

1Elementary functions

$$(a)\ (-1)^{1/3} \quad (b)\ \sqrt{2-i} \quad (c)\ (-1-i)^{-i/2}$$

Some Elementary Functions - Write answers in form $a+bi$

2-Taylor series

Find the Taylor series expansion for $f(z) = \ln(1+z)$

3-Taylor series

Use known Taylor series expansions about the origin to determine

$$f(z) = \frac{\sin z}{1-z}$$

4-Taylor and Laurent expansions

For each function, find all the Taylor series and Laurent series expansions about the point $z = a$ and state the region of convergence

$$(a)\ \frac{1}{z}\ \ a=1 \quad (b)\ \frac{1}{z^2+1}\ \ a=i \quad (c)\ \frac{1}{(z+1)(z-2)}\ \ a=2$$

5-Residues

Find the residue of each function at each pole.

$$(a)\ \frac{1}{z^2}\sin 2z \quad (b)\ \frac{\cos z}{z^2+2z+1}$$

6-Integrals around a circle

Evaluate each integral around the circle $|z| = 2$.

$$(a)\ \oint \frac{\sin z}{z^3}dz \quad (b)\ \oint \frac{\sin z}{z^3-z^2}dz \quad (c)\ \oint \frac{z^2+1}{z(z+1)^3}dz$$

7-Real Integrals

Determine the value of each real integral

$$(a)\ \int_0^{2\pi} \frac{d\theta}{(2+cos\theta)^2} \quad (b)\ \int_0^{2\pi} \frac{\sin 2\theta d\theta}{(5+4cos\theta)}$$

8-Evaluate integrals

Evaluate each integral.

$$(a)\ \int_{-\infty}^{\infty} \frac{1+x}{1+x^3}dx \quad (b)\ \int_{-\infty}^{\infty} \frac{\cos x}{(1+x^2)^2}dx \quad (c)\ \int_{-\infty}^{\infty} \frac{\cos x}{1+x^4}dx$$

9- Value of integral

Find the value of

$$\int_{-\infty}^{\infty} \frac{1}{x^4-1}dx$$

using a closed path where you have a large semicircle in the upper half-plane and you integrate around(small semicircles in upper half-plane) the two poles on the real axis.

Assignment #11-Tensors

1-Is it a tensor?

Use the basic definition of a Cartesian tensor to show the following.

(a) That for any general, but fixed, ϕ,

$$(u_1, u_2) = (x_1 \cos\phi - x_2 \sin\phi, x_1 \sin\phi + x_2 \cos\phi)$$

are the components of a first-order tensor in two dimensions.

(b) That

$$\begin{pmatrix} x_2^2 & x_1 x_2 \\ x_1 x_2 & x_1^2 \end{pmatrix}$$

is not a tensor of order 2. To establish that a single element does not transform correctly is sufficient.is not a tensor of order 2. To establish that a single element does not trans-form correctly is sufficient.

2-Component of tensor in new system

The components of two vectors **A** and **B** and a second-order tensor **T** are given in one coordinate system by

$$\mathbf{A} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \;,\; \mathbf{B} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \;,\; \mathbf{T} = \begin{pmatrix} 2 & \sqrt{3} & 0 \\ \sqrt{3} & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

n a second coordinate system, obtained from the first by rotation, the components of **A** and **B** are

$$\mathbf{A}' = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 0 \\ 1 \end{pmatrix} \;,\; \mathbf{B}' = \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ \sqrt{3} \end{pmatrix}$$

Find the components of **T** in this new coordinate system and hence evaluate, with a minimum of calculation,

$$T_{ij}T_{ji} \;,\; T_{ki}T_{jk}T_{ij} \;,\; T_{ik}T_{mn}T_{ni}T_{km}$$

3-Quotient law

Use the quotient law for tensors to show that the array

$$\begin{pmatrix} y^2+z^2-x^2 & -2xy & -2xz \\ -2yx & x^1+z^2-y^2 & -2yz \\ -2zx & -2zy & x^2+y^2-z^2 \end{pmatrix}$$

forms a second-order tensor.

4-Why it must be so

A column matrix **a** has components a_x, a_y, a_z and **A** is the matrix with elements $A_{ij} = -\epsilon_{ijk}a_k$.

(a) What is the relationship between column matrices **b** and **c** if **Ab** = **c**?

(b) Find the eigenvalues of **A** and show that **a** is one of its eigenvectors. Explain why this must be so.

5-Second covariant derivative

Find an expression for the second covariant derivative, written in semi-colon notation as $v_{i;jk} \equiv (v_{i;j})_{;k}$, of a vector v_i. By interchanging the order of differentiation and then subtracting the two expressions, we define the components R^l_{ijk} of the Riemann tensor as

$$v_{i;jk} - v_{i;kj} \equiv R^l_{ijk} v_l$$

Show that in a general coordinate system u^i these components are given by

$$R^l_{ijk} = \frac{\partial \Gamma^l_{ik}}{\partial u^j} - \frac{\partial \Gamma^l_{ij}}{\partial u^k} + \Gamma^m_{ik}\Gamma^l_{mj} - \Gamma^m_{ij}\Gamma^l_{mk}$$

By first considering Cartesian coordinates, show that all the components $R^l_{ijk} = 0$ for any coordinate system in three-dimensional Euclidean space.

In such a space, therefore, we may change the order of the covariant derivatives without changing the resulting expression.

6-Christoffel symbol

We may define Christoffel symbols of the first kind by

$$\Gamma_{ijk} = g_{il}\Gamma^l_{jk}$$

Show that these are given by

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{ki}}{\partial u^j} + \frac{\partial g_{ij}}{\partial u^k} - \frac{\partial g_{jk}}{\partial u^i}\right)$$

By permuting indices, verify that

$$\frac{\partial g_{ij}}{\partial u^k} = \Gamma_{ijk} + \Gamma_{jik}$$

Using the fact that $\Gamma^l_{jk} = \Gamma^l_{kj}$ show that

$$g_{ij;k} \equiv 0$$

i.e. that the covariant derivative of the metric tensor is identically zero in all coordinate systems.

7-Parabolic coordinates

Paraboloidal coordinates u, v, ϕ are defined in terms of Cartesian coordinates by

$$x = uv\cos\phi \ \ , \ \ y = uv\sin\phi \ \ , \ \ z = \frac{1}{2}(U^2 - v^2)$$

Identify the coordinate surfaces in the u, v, ϕ system. Verify that each coordinate surface (u = constant, say) intersects every coordinate surface on which one of the other two coordinates (v, say) is constant. Show further that the system of coordinates is an orthogonal one and determine its scale factors. Prove that the u-component of $\nabla \times \mathbf{a}$ is given by

$$\frac{1}{(u^2+v^2)^{1/2}}\left(\frac{a_\phi}{v} - \frac{\partial a_\phi}{\partial v}\right) - \frac{1}{uv}\frac{\partial a_v}{\partial \phi}$$

8-Fourth-order tensor

A fourth-order tensor T_{ijkl} has the properties

$$T_{jikl} = -T_{ijkl} \quad , \quad T_{ijlk} = -T_{ijkl}$$

Prove that for any such tensor there exists a second-order tensor K_{mn} such that

$$T_{ijkl} = \epsilon_{ijm}\epsilon_{kln}K_{mn}$$

and give an explicit expression for K_{mn}. Consider two (separate) special cases, as follows.

(a) A tensor which has the same components in all rotated coordinate systems. All rank-0 tensors (scalars) are isotropic, but no rank-1 tensors (vectors) are. The unique rank-2 isotropic tensor is the Kronecker delta, and the unique rank-3 isotropic tensor is the permutation symbol. Given that T_{ijkl} is isotropic and $T_{ijji} = 1$ show that T_{ijkl} is uniquely determined and express it in terms of Kronecker deltas.

(b) If now T_{ijkl} has the additional property

$$T_{klij} = -T_{ijkl}$$

show that T_{ijkl} has only three linearly independent components and find an expression for T_{ijkl} in terms of thew vector

$$V_i = -\frac{1}{4}\epsilon_{jkl}T_{ijkl}$$

9-Covariant vector

Using the fact that ds^2 is invariant, prove that g_{ij} is a second-order covariant tensor.

Assignment #12-Probability and Statistics

1-Give advice?

A and B each have two unbiased four-faced dice, the four faces being numbered 1, 2, 3 and 4. Without looking, B tries to guess the sum x of the numbers on the bottom faces of A?s two dice after they have been thrown onto a table. If the guess is correct B receives x^2 euros, but if not he loses x euros.

Determine B?s expected gain per throw of A's dice when he adopts each of the following strategies:

(a) he selects x at random in the range $2 \leq x \leq 8$;

(b) he throws his own two dice and guesses x to be whatever they indicate;

(c) he takes your advice and always chooses the same value for x. Which number would you advise?

2-Two duelists

Two duellists, A and B, take alternate shots at each other, and the duel is over when a shot (fatal or otherwise!) hits its target. Each shot fired by A has a probability α of hitting B, and each shot fired by B has a probability β of hitting A. Calculate the probabilities P_1 and P_2, defined as follows, that A will win such a duel: P_1, A fires the first shot; P_2, B fires the first shot.

If they agree to fire simultaneously, rather than alternately, what is the probability P_3that A will win, i.e., hit B without being hit himself?

3-Defective chips

An electronics assembly firm buys its microchips from three different suppliers; half of them are bought from firm X, whilst firms Y and Z supply 30% and 20%, respectively. The suppliers use different quality-control procedures and the percentages of defective chips are 2%, 4% and 4% for X, Y and Z, respectively. The probabilities that a defective chip will fail two or more assembly-line tests are 40%, 60% and 80%, respectively, whilst all defective chips have a 10% chance of escaping detection. An assembler finds a chip that fails only one test. What is the probability that it came from supplier X?

4-Odds in dice rolling

This exercise shows that the odds are hardly ever ?'evens' when it comes to dice rolling.

(a) Gamblers A and B each roll a fair six-faced die, and B wins if his score is strictly greater than A's. Show that the odds are 7 to 5 in A's favor.

(b) Calculate the probabilities of scoring a total T from two rolls of a fair die for T = 2,3,...,12. Gamblers C and D each roll a fair die twice and score respective totals T_C and T_D , D winning if $T_D > T_C$. Realizing that the odds are not equal, D insists that C should increase her stake for each game. C agrees to stake \$1.10 per game, as compared to D's \$1.00 stake. Who will show a profit?

5-Kittens

Kittens from different litters do not get on with each other and fighting breaks out whenever two kittens from different litters are present together. A cage initially contains x kittens from one litter and y from another. To quell the fighting, kittens are removed at random, one at a time, until peace is restored. Show, by induction, that the expected number of kittens finally remaining is

$$N(x,y) = \frac{x}{y+1} + \frac{y}{x+1}$$

6-Lottery

As assistant to a celebrated and imperious newspaper proprietor, you are given the job of running a lottery in which each of his five million readers will have an equal independent chance p of winning a million pounds; you have the job of choosing p. However, if nobody wins it will be bad for publicity, whilst, if more than two readers do so, the prize cost will more than offset the profit from extra circulation - in either case you will be sacked! Show that, however you choose p, there is more than a 40% chance you will soon be clearing your desk.

7-Family

A husband and wife decide that their family will be complete when it includes two boys and two girls - but that this would then be enough! The probability that a new baby will be a girl is p. Ignoring the possibility of identical twins, show that the expected size of their family is

$$2\left(\frac{1}{pq} - 1 - pq\right)$$

where $q = 1 - p$.

8-A circle

A point P is chosen at random on the circle $x^2 + y^2 = 1$. The random variable X denotes the distance of P from (1,0). Find the mean and variance of X and the probability that X is greater than its mean.

9-Two cards

Two cards are drawn at random from a shuffled deck and laid aside without being examined. The a third card is drawn. Show that the probability that the third card is a spade is 1/4 just as it was for the first card. HINT: Consider all the *mutually exclusive) possibilities (two discarded cards spades, third card spade or not spade, etc).

10-Coins

Suppose you have 3 nickels and 4 dimes in your right pocket and 2 nickels and a quarter in your left pocket. You pick a pocket at random and from it select a coin at random. If it is a nickel, what is the probability that it came from your right pocket?

11-Misprints

If there are 100 misprints in a magazine of 40 pages, on how many pages would you expect to find no misprints? Two misprints? Five misprints? (Poisson distribution).

Assignment #13-Group Theory and Elementary Particle

1-Binary operation

Define a binary operation $\circ$ on the set of real numbers by

$$x \circ y = x + y + rxy$$

where r is a non-zero real number. Show that the operation $\circ$ is associative.

Prove that $x \circ y = -r^{-1}$ if, and only if, $x = -r^{-1}$ or $y = -r^{-1}$. Hence prove that the set of all real numbers excluding $-r^{-1}$ forms a group under the operation $\circ$.

2-Is it a group?

$\mathcal{S}$ is the set of all 2×2 matrices of the form

$$A = \begin{pmatrix} W & x \\ y & z \end{pmatrix} \qquad \text{where } wz - xy = 1$$

Show that $\mathcal{S}$ is a group under matrix multiplication. Which element(s) have order 2? Prove that an element A has order 3 if $w + z + 1 = 0$. Order of element $a \in G$ is the smallest positive integer n, such that $a^n = e$, where e denotes the identity element of the group, and a^n denotes the product of n copies of a.

3-An Abelian group

Show that if p is prime then the set of rational number pairs (a, b), excluding $(0, 0)$, with multiplication defined by

$$(a,) \circ (c, d) = (e, f) \quad \text{where} \quad (a + b\sqrt{p})(c + d\sqrt{p}) = e + f\sqrt{p}$$

forms an Abelian group.