# 10 - Potentials and Fields

## 10.1 THE POTENTIAL FORMULATION

### 10.1.1 Scalar and Vector Potentials

In this chapter we seek the *general* solution to Maxwell's equations,

$$\left.\begin{array}{llll}\text{(i)} & \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho, & \text{(iii)} & \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}, \\ \text{(ii)} & \nabla \cdot \mathbf{B} = 0, & \text{(iv)} & \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t}.\end{array}\right\} \tag{10.1}$$

Given $\varrho(\mathbf{r}, t)$ and $\mathbf{J}(\mathbf{r}, t)$, what are the fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$?

In the static case, Coulomb's law and the Biot-Savart law provide the answer.

What we're looking for, then, is the generalization of those laws to time-dependent configurations.

This is not an easy problem, and it pays to begin by representing the fields in terms of potentials.

In electrostatics $\nabla \times \mathbf{E} = \mathbf{0}$ allowed us to write $\mathbf{E}$ as the gradient of a scalar potential: $\mathbf{E} = -\nabla V$ .

In electro*dynamics* this is no longer possible, because the curl of $\mathbf{E}$ is nonzero.

But $\mathbf{B}$ remains divergenceless, so we can still write

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A},} \tag{10.2}$$

as in magnetostatics.

Putting this into Faraday's law (iii) yields

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}),$$

or

$$\nabla \times \left( \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = \mathbf{0}.$$

*Here* is a quantity, unlike **E** alone, whose curl *does* vanish; it can therefore be written as the gradient of a scalar:

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V.$$

In terms of $V$ and **A**, then,

$$\boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}.} \tag{10.3}$$

This reduces to the old form, of course, when **A** is constant.

The potential representation (Eqs. 10.2 and 10.3) automatically fulfills the two homogeneous Maxwell equations, (ii) and (iii).

How about Gauss's law (i) and the Ampère/Maxwell law (iv)?

Putting Eq. 10.3 into (i), we find that

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{1}{\epsilon_0}\rho; \tag{10.4}$$

this replaces Poisson's equation (to which it reduces in the static case).

Putting Eqs. 10.2 and 10.3 into (iv) yields

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J} - \mu_0 \epsilon_0 \nabla \left( \frac{\partial V}{\partial t} \right) - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2},$$

or, using the vector identity $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$, and rearranging the terms a bit:

$$\left(\nabla^2\mathbf{A} - \mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2}\right) - \nabla\left(\nabla\cdot\mathbf{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\mathbf{J}. \tag{10.5}$$

Equations 10.4 and 10.5 contain all the information in Maxwell's equations.

**Example 10.1.** Find the charge and current distributions that would give rise to the potentials

$$V = 0, \quad \mathbf{A} = \begin{cases} \dfrac{\mu_0 k}{4c}(ct - |x|)^2\,\hat{\mathbf{z}}, & \text{for } |x| < ct, \\ \mathbf{0}, & \text{for } |x| > ct, \end{cases}$$

where $k$ is a constant, and (of course) $c = 1/\sqrt{\epsilon_0\mu_0}$ .

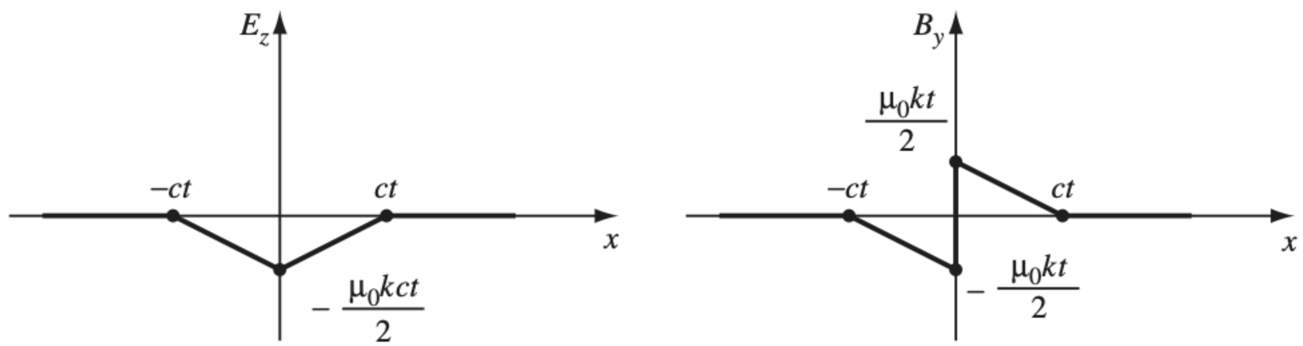


**FIGURE 10.1**

## Solution

First we'll determine the electric and magnetic fields, using Eqs. 10.2 and 10.3:

$$\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} = -\frac{\mu_0 k}{2}(ct - |x|)\,\hat{\mathbf{z}},$$

$$\mathbf{B} = \nabla\times\mathbf{A} = -\frac{\mu_0 k}{4c}\frac{\partial}{\partial x}(ct - |x|)^2\,\hat{\mathbf{y}} = \pm\frac{\mu_0 k}{2c}(ct - |x|)\,\hat{\mathbf{y}},$$

(plus, for $x > 0$; minus, for $x < 0$).

These are for $|x| < ct$; when $|x| > ct$, $\mathbf{E} = \mathbf{B} = \mathbf{0}$ (Fig. 10.1).

Calculating every derivative in sight, I find

$$\nabla \cdot \mathbf{E} = 0; \quad \nabla \cdot \mathbf{B} = 0; \quad \nabla \times \mathbf{E} = \mp \frac{\mu_0 k}{2} \hat{\mathbf{y}}; \quad \nabla \times \mathbf{B} = -\frac{\mu_0 k}{2c} \hat{\mathbf{z}};$$

$$\frac{\partial \mathbf{E}}{\partial t} = -\frac{\mu_0 k c}{2} \hat{\mathbf{z}}; \quad \frac{\partial \mathbf{B}}{\partial t} = \pm \frac{\mu_0 k}{2} \hat{\mathbf{y}}.$$

As you can easily check, Maxwell's equations are all satisfied, with $\varrho$ and $\mathbf{J}$ both *zero*.

Notice, however, that $\mathbf{B}$ has a discontinuity at $x = 0$, and this signals the presence of a surface current $\mathbf{K}$ in the $yz$ plane; boundary condition (iv) in Eq. 7.64 gives

$$kt\, \hat{\mathbf{y}} = \mathbf{K} \times \hat{\mathbf{x}},$$

and hence

$$\mathbf{K} = kt\, \hat{\mathbf{z}}.$$

Evidently we have here a uniform surface current flowing in the $z$ direction over the plane $x = 0$, which starts up at $t = 0$, and increases in proportion to $t$.

Notice that the news travels out (in both directions) at the speed of light: for points $|x| > ct$ the message ("current is now flowing") has not yet arrived, so the fields are zero.

### 10.1.2 Gauge Transformations

Equations 10.4 and 10.5 are *ugly*, and you might be inclined to abandon the potential formulation altogether.

However, we *have* succeeded in reducing six problems—finding $\mathbf{E}$ and $\mathbf{B}$ (three components each)—down to four:

$V$ (one component) and $\mathbf{A}$ (three more).

Moreover, Eqs. 10.2 and 10.3 do not uniquely define the potentials; we are free to impose extra conditions on $V$ and **A**, as long as nothing happens to **E** and **B**.

Let's work out precisely what this **gauge freedom** entails.

Suppose we have two sets of potentials, $(V, \mathbf{A})$ and $(V', \mathbf{A}')$, which correspond to the *same* electric and magnetic fields.

By how much can they differ?

Write

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\alpha} \quad \text{and} \quad V' = V + \beta.$$

Since the two **A**'s give the same **B**, their curls must be equal, and hence

$$\nabla \times \boldsymbol{\alpha} = \mathbf{0}.$$

We can therefore write α as the gradient of some scalar:

$$\boldsymbol{\alpha} = \nabla \lambda.$$

The two potentials also give the same **E**, so

$$\nabla \beta + \frac{\partial \boldsymbol{\alpha}}{\partial t} = \mathbf{0},$$

or

$$\nabla \left( \beta + \frac{\partial \lambda}{\partial t} \right) = \mathbf{0}.$$

The term in parentheses is therefore independent of position (it could, however, depend on time); call it $k(t)$:

$$\beta = -\frac{\partial \lambda}{\partial t} + k(t).$$

Actually, we might as well absorb $k(t)$ into $\lambda$, defining a new $\lambda$ by adding $\int_0^t k(t')dt'$ to the old one.

This will not affect the gradient of $\lambda$; it just adds $k(t)$ to $\partial\lambda/\partial t$.

It follows that

$$\left.\begin{aligned} \mathbf{A}' &= \mathbf{A} + \nabla\lambda, \\ V' &= V - \frac{\partial\lambda}{\partial t}. \end{aligned}\right\} \tag{10.7}$$

**Conclusion:** For any old scalar function $\lambda(\mathbf{r}, t)$, we can with impunity add $\nabla\lambda$ to $\mathbf{A}$, provided we simultaneously subtract $\partial\lambda/\partial t$ from $V$.

This will not affect the physical quantities **E** and **B**. Such changes in $V$ and **A** are called **gauge transformations**.

They can be exploited to adjust the divergence of **A**, with a view to simplifying the "ugly" equations 10.4 and 10.5.

In magnetostatics, it was best to choose $\nabla \cdot \mathbf{A} = 0$ (Eq. 5.63); in electrodynamics, the situation is not so clear cut, and the most convenient gauge depends to some extent on the problem at hand.

There are many famous gauges in the literature; I'll show you the two most popular ones.

### 10.1.3 Coulomb Gauge and Lorenz Gauge

**The Coulomb Gauge**.

As in magnetostatics, we pick

$$\nabla \cdot \mathbf{A} = 0. \tag{10.8}$$

With this, Eq. 10.4 becomes

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho. \tag{10.9}$$

This is Poisson's equation, and we already know how to solve it: setting $V = 0$ at infinity,

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t)}{\mathscr{r}} \, d\tau'. \tag{10.10}$$

There is a very peculiar thing about the scalar potential in the Coulomb gauge: it is determined by the distribution of charge *right now*.

If I move an electron in my laboratory, the potential $V$ on the moon immediately records this change.

That sounds particularly odd in the light of special relativity, which allows no message to travel faster than $c$.

The point is that $V$ *by itself* is not a physically measurable quantity—all the man in the moon can measure is **E**, and that involves **A** as well (Eq. 10.3).

Somehow it is built into the vector potential (in the Coulomb gauge) that whereas $V$ instantaneously reflects all changes in $\varrho$, the combination $-\nabla V - (\partial \mathbf{A}/\partial t)$ does *not*; **E** will change only after sufficient time has elapsed for the "news" to arrive.

The *advantage* of the Coulomb gauge is that the scalar potential is particularly simple to calculate; the *disadvantage* (apart from the acausal appearance of $V$) is that **A** is particularly *difficult* to calculate.

The differential equation for **A** (Eq. 10.5) in the Coulomb gauge reads

$$\nabla^2 \mathbf{A} - \mu_0\epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} + \mu_0\epsilon_0 \nabla \left( \frac{\partial V}{\partial t} \right). \tag{10.11}$$

**The Lorenz gauge**.

In the Lorenz gauge, we pick

$$\boxed{\nabla \cdot \mathbf{A} = -\mu_0\epsilon_0 \frac{\partial V}{\partial t}.} \tag{10.12}$$

This is designed to eliminate the middle term in Eq. 10.5.

With this,

$$\nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}. \tag{10.13}$$

Meanwhile, the differential equation for $V$, (Eq. 10.4), becomes

$$\nabla^2 V - \mu_0 \epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{1}{\epsilon_0} \rho. \tag{10.14}$$

The virtue of the Lorenz gauge is that it treats $V$ and $\mathbf{A}$ on an equal footing: the same differential operator

$$\boxed{\nabla^2 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \equiv \Box^2,} \tag{10.15}$$

(called the **d'Alembertian**) occurs in both equations:

$$\boxed{\begin{aligned} &\text{(i)} \quad \Box^2 V = -\frac{1}{\epsilon_0} \rho, \\ &\text{(ii)} \quad \Box^2 \mathbf{A} = -\mu_0 \mathbf{J}. \end{aligned}} \tag{10.16}$$

This democratic treatment of $V$ and $\mathbf{A}$ is especially nice in the context of special relativity, where the d'Alembertian is the natural generalization of the Laplacian, and Eqs. 10.16 can be regarded as four-dimensional versions of Poisson's equation.

In this same spirit, the wave equation $\Box^2 f = 0$, might be regarded as the four-dimensional version of Laplace's equation.

In the Lorenz gauge, $V$ and $\mathbf{A}$ satisfy the **inhomogeneous wave equation**, with a “source” term (in place of zero) on the right.

From now on, I shall use the Lorenz gauge exclusively, and the whole of electrodynamics reduces to the problem of *solving the inhomogeneous wave equation for a specified source*.

### 10.1.4 Lorentz Force Law in Potential Form

It is illuminating to express the Lorentz force law in terms of potentials:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = q\left[-\nabla V - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times (\nabla \times \mathbf{A})\right], \tag{10.17}$$

where $\mathbf{p} = m\mathbf{v}$ is the momentum of the particle.

Now, product rule 4 says

$$\nabla(\mathbf{v} \cdot \mathbf{A}) = \mathbf{v} \times (\nabla \times \mathbf{A}) + (\mathbf{v} \cdot \nabla)\mathbf{A}$$

($\mathbf{v}$, the velocity of the particle, is a function of time, but not of position).

Thus

$$\frac{d\mathbf{p}}{dt} = -q\left[\frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{A} + \nabla(V - \mathbf{v} \cdot \mathbf{A})\right]. \tag{10.18}$$

The combination

$$\left[\frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{A}\right]$$

is called the **convective derivative** of $\mathbf{A}$, and written $d\mathbf{A}/dt$ (*total* derivative).

It represents the time rate of change of $\mathbf{A}$ *at the (moving) location of the particle*.

For suppose that at time $t$ the particle is at point $\mathbf{r}$, where the potential is $\mathbf{A}(\mathbf{r},t)$; a moment $dt$ later it is at $\mathbf{r} + \mathbf{v}dt$, where the potential is $\mathbf{A}(\mathbf{r} + \mathbf{v}dt, t + dt)$.

The *change* in $\mathbf{A}$, then, is

$$\begin{aligned} d\mathbf{A} &= \mathbf{A}(\mathbf{r}+\mathbf{v}\,dt, t+dt) - \mathbf{A}(\mathbf{r}, t) \\ &= \left(\frac{\partial \mathbf{A}}{\partial x}\right)(v_x\,dt) + \left(\frac{\partial \mathbf{A}}{\partial y}\right)(v_y\,dt) + \left(\frac{\partial \mathbf{A}}{\partial z}\right)(v_z\,dt) + \left(\frac{\partial \mathbf{A}}{\partial t}\right)dt, \end{aligned}$$

so

$$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{A}. \tag{10.19}$$

As the particle moves, the potential it "feels" changes for two distinct reasons: first, because the potential varies with *time*, and second, because it is now in a new location, where **A** is different because of its variation in *space*.

Hence the two terms in Eq. 10.19.

With the aid of the convective derivative, the Lorentz force law reads:

$$\frac{d}{dt}(\mathbf{p} + q\mathbf{A}) = -\nabla\left[q(V - \mathbf{v}\cdot\mathbf{A})\right]. \tag{10.20}$$

This is reminiscent of the standard formula from mechanics, for the motion of a particle whose potential energy $U$ is a specified function of position:

$$\frac{d\mathbf{p}}{dt} = -\nabla U.$$

Playing the role of **p** is the so-called **canonical momentum**,

$$\mathbf{p}_{\text{can}} = \mathbf{p} + q\mathbf{A}, \tag{10.21}$$

while the part of $U$ is taken by the velocity-dependent quantity

$$U_{\text{vel}} = q(V - \mathbf{v}\cdot\mathbf{A}). \tag{10.22}$$

A similar argument gives the rate of change of the particle's *energy*:

$$\frac{d}{dt}(T + qV) = \frac{\partial}{\partial t}[q(V - \mathbf{v} \cdot \mathbf{A})], \tag{10.23}$$

where $T = {}^1\!/_2\, m\mathrm{v}^2$ is its kinetic energy and $qV$ is its potential energy (The derivative on the right acts only on $V$ and $\mathbf{A}$, not on $\mathbf{v}$).

Curiously, the same quantity $U_{\text{vel}}$ appears on the right side of both equations.

The parallel between Eq. 10.20 and Eq. 10.23 invites us to interpret $\mathbf{A}$ as a kind of "potential momentum" per unit charge, just as $V$ is potential *energy* per unit charge.