a

We are given that

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{1}$$

So, it is immediately evident that $A^\dagger = A$ because A is real and is its own transpose. Furthermore, $AA^\dagger = I$ so A is also unitary. To find the eigenvalues, set the determinant equal to zero which gives

$$-\lambda^2(1+\lambda) + (1+\lambda) = (1+\lambda)(1-\lambda^2) = 0 \tag{2}$$

Therefore the eigenvalues are

$$\lambda = 1, -1, -1 \tag{3}$$

Plugging in $\lambda_1 = 1$ and doing some row reduction gives

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4}$$

which gives the eigenvector, after normalization, of

$$\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \tag{5}$$

Plugging in $\lambda_2 = \lambda_3 = -1$ and doing row reduction gives

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6}$$

Since this eigenvalue is degenerate, we are looking for two orthonormal eigenvectors that are both solutions. The first we find the usual way as

$$\vec{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{7}$$

Then looking at this eigenvector and at the corresponding matrix, it is easy to find a solution that is orthonormal, namely

$$\vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{8}$$

It is easy to verify that $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are orthonormal.

b

Now we find the projection operator matrices by computing

$$\hat{P}_1 = |\vec{v}_1\rangle \langle\vec{v}_1| \tag{9}$$

$$\hat{P}_2 = |\vec{v}_2\rangle \langle\vec{v}_2| \tag{10}$$

$$\hat{P}_3 = |\vec{v}_3\rangle \langle\vec{v}_3| \tag{11}$$

So in matrix form this gives

$$P_1 = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \tag{12}$$

$$P_2 = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{13}$$

$$P_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{14}$$

In the world of hard/soft particles, let us define a state using a hardness basis $|h\rangle, |s\rangle$, where $\hat{O}_{HARDNESS}|h\rangle = |h\rangle, \hat{O}_{HARDNESS}|s\rangle = -|s\rangle$ and the hardness operator $\hat{O}_{HARDNESS}$ is represented by

$$\hat{O}_{HARDNESS} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In the state, $|A\rangle = \cos\theta |h\rangle + e^{i\phi}\sin\theta |s\rangle$. This state is normalized because...

$$\langle A| |A\rangle = \cos^2\theta + e^{i\phi}e^{-i\phi}\sin^2\theta = 1 \tag{1}$$

For two states, A and B to be orthogonal, $\langle A| |B\rangle = 0$. The state $|B\rangle$ is also normalized therefore, $| |B\rangle |^2 = |a|^2 + |b|^2 = 1$. Given $|B\rangle = a|h\rangle + b|s\rangle$ we want to solve for a and b so that $|B\rangle$ satisfies the above conditions.

So,

$$\langle A| |B\rangle = a\cos\theta |s\rangle + b\sin\theta e^{-i\phi} |h\rangle = 0 \tag{2}$$

Therefore,

$$a = e^{-i\phi}\sin\theta \tag{3}$$

$$b = -\cos\theta \tag{4}$$

$$|B\rangle = e^{-i\phi}\sin\theta |h\rangle - \cos\theta |s\rangle \tag{5}$$

Next, I can solve for $|h\rangle$ and $|s\rangle$ in terms of $|A\rangle$ and $|B\rangle$.

$$|h\rangle = \cos\theta |A\rangle - e^{i\phi}\sin\theta |B\rangle \tag{6}$$

$$|s\rangle = e^{-i\phi}\sin\theta |A\rangle - \cos\theta |B\rangle \tag{7}$$

The possible outcomes of hardness measurement on state $|A\rangle$ are h and s. The probablities that they will occur are...

$$P(h|A) = \cos^2\theta \tag{8}$$

$$P(s|A) = sin^2\theta \tag{9}$$

The hardness operator in the $|A\rangle, |B\rangle$ basis is,

$$\hat{O}_{HARDNESS} = \begin{pmatrix} \cos\theta & -e^{i\phi}\sin\theta \\ e^{-i\phi}\sin\theta & \cos\theta \end{pmatrix} \tag{10}$$

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Physics 113
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In this problem we are given the matrix

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

,

and asked to find the eigenvalues and corresponding eigenvectors. Then, we want to construct the corresponding projection operators and verify that the matrix can be written in terms of its eigenvalues and eigenvectors.

First, to find the eigenvalues we have:

$$\det(M) = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0.$$

This gives us the eigenvalues of:

$$-\lambda(\lambda^2 - 1) - 1(-\lambda) = 0$$
$$\lambda^3 - 2\lambda = 0$$
$$\lambda(\lambda^2 - 2) = 0$$
$$\lambda_1 = 0, \lambda_2 = \sqrt{2}, \lambda_3 = -\sqrt{2}.$$

We are then able to find eigenvectors of

$$\vec{v_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

,

$$\vec{v_2} = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

, and

$$\vec{v_3} = \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

.

Then, using these vectors, we have the projection operators of:

$$P_1 = |1\rangle\langle 1| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

,

$$P_2 = |2\rangle\langle 2| = \frac{1}{4}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 2 & \sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix}$$

,

$$P_3 = |3\rangle\langle 3| = \frac{1}{4}\begin{pmatrix} 1 & -\sqrt{2} & 1 \\ -\sqrt{2} & 2 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$$

.

Then, to verify that the matrix can be written in terms of its eigenvalues and eigenvectors, we multiply the eigenvectors by their corresponding projection matrix and add all of these together. This addition should equal the original matrix.

So,

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \frac{\sqrt{2}}{4}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 2 & \sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix} - \frac{\sqrt{2}}{4}\begin{pmatrix} 1 & -\sqrt{2} & 1 \\ -\sqrt{2} & 2 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$$

,

B.4.22.16

Let

$$R = \begin{pmatrix} 6 & -2 \\ -2 & 9 \end{pmatrix}$$

represent an observable, and

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

be an arbitrary state vector (with $|a|^2 + |b|^2 = 1$). Calculate $\langle R^2 \rangle$ in two ways:

(a) Evaluate $\langle R^2 \rangle = \langle \psi | R^2 | \psi \rangle$ directly.

(b) Find the eigenvalues (r_1 and r_2) and eigenvectores ($|r_1\rangle$ and $|r_2\rangle$) of R^2 or R. Expand the state vector as a linear combination of eigenvectors and evaluate.

$$\langle R^2 \rangle = r_1^2 |c_1|^2 + r_2^2 |c_2|^2$$

Solution.a

$$\langle R^2 \rangle = \langle \psi | R^2 | \psi \rangle = \begin{pmatrix} a^* & b^* \end{pmatrix} \begin{pmatrix} 40 & -30 \\ -30 & 85 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tag{1}$$

$$= 40a^*a - 30ab^* - 30a^*b + 85b^*b \tag{2}$$

Solution.b

The Characteristic polynomial

$$(6 - \lambda)(9 - \lambda) - 4 = (\lambda - 5)(\lambda - 10) = 0 \tag{3}$$

yields the eigenvalues

$$\lambda = 5, 10. \tag{4}$$

The corresponding eigenvectors are then

$$|5\rangle = \ker \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad |10\rangle = \ker \begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ -2 \end{pmatrix}. \tag{5}$$

Because R^2 is hermitian, these eigenvectors are orthogonal. We want to write $|\psi\rangle$ in terms of these vectors, that is, find c_5 and c_{10} such that

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} = c_5|5\rangle + c_{10}|10\rangle. \tag{6}$$

We can solve the system

$$\frac{1}{\sqrt{5}} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c_5 \\ c_{10} \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{7}$$

which yields

$$c_5 = \frac{1}{\sqrt{5}}(b + 2a) \qquad c_{10} = \frac{1}{\sqrt{5}}(2b - a). \tag{8}$$

The expectation value of R^2 can now be written as

$$\langle\psi|R^2|\psi\rangle = \Big(c_5^*\langle 5| + c_{10}^*\langle 10|\Big)\hat{R}^2\Big(c_5|5\rangle + c_{10}|10\rangle\Big) \tag{9}$$

$$= \Big(c_5\langle 5| + c_{10}\langle 10|\Big)\Big(25c_5|5\rangle + 100c_{10}|10\rangle\Big) = 25|c_1|^2 + 100|c_2|^2. \tag{10}$$

Plugging the coefficients we found in equation 8, we get

$$\langle R^2\rangle = 40a^*a - 30ab^* - 30a^*b + 85b^*b, \tag{11}$$

which is consistent with out solution in part a.

B.4.22.19

For all parts of this problem, let $\mathcal{H}$ be a Hilbert space spanned by the basis kets $\{|0\rangle, |1\rangle, |2\rangle|3\rangle\}$, and let a and b be arbitrary complex constants

(a) Which of the following are Hermitian operators on $\mathcal{H}$?

1. $|0\rangle\langle 1| + i|1\rangle\langle 0|$
2. $|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 3| + |3\rangle\langle 2|$
3. $(a|0\rangle + |1\rangle)^\dagger(a|0\rangle + |1\rangle)$
4. $((a|0\rangle + b^*|1\rangle)^\dagger(b|0\rangle - a^*|1\rangle))|2\rangle\langle 1| + |3\rangle\langle 3|$
5. $|0\rangle\langle 0| + i|1\rangle\langle 0| - i|0\rangle\langle 1| + |1\rangle\langle 1|$

(b) Find the spectral decomposition of the following operator on $\mathcal{H}$:

$$\hat{K} = |0\rangle\langle 0|_2|1\rangle\langle 2| + 2|2\rangle\langle 1| - |3\rangle\langle 3|$$

(c) Let $|\psi\rangle$ be a normalized ket in $\mathcal{H}$, and let $\hat{I}$ denote the identity operator on $\mathcal{H}$. Is the operator

$$\hat{B} = \tfrac{1}{\sqrt{2}}\left(\hat{I} + |\psi\rangle\langle\psi|\right)$$

a projection operator?

(d) Find the spectral decomposition of the operator $\hat{B}$ from part (c).

Solution.a

1. Not hermitian

$$\hat{P}_1^\dagger = |1\rangle\langle 0| - i|0\rangle\langle 1| \neq \hat{P}_1^\dagger \tag{12}$$

2. Hermitian

$$\hat{P}_2^\dagger = |0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 3| + |3\rangle\langle 2| = \hat{P}_2 \tag{13}$$

3. Hermitian

$$\hat{P}_3 = (a|0\rangle + |1\rangle)^\dagger(a|0\rangle + |1\rangle) = a^*a + 1 = \text{a real number} \tag{14}$$

Real numbers are hermitian.

4. Hermitian

$$\hat{P}_4 = \Big((a|0\rangle + b^*|1\rangle)^\dagger (b|0\rangle - a^*|1\rangle)\Big)|2\rangle\langle 1| + |3\rangle\langle 3| \tag{15}$$

$$= (a^*b - a^*b)|2\rangle\langle 1| + |3\rangle\langle 3| = |3\rangle\langle 3| \tag{16}$$

Projection operators are hermitian.

5. Hermitian

$$\hat{P}_5^\dagger = |0\rangle\langle 0| - i|0\rangle\langle 1| + i|1\rangle\langle 0| + |1\rangle\langle 1| = \hat{P}_5 \tag{17}$$

Solution.b

$$\hat{K} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{18}$$

We can pick out 3 smaller matrices along the diagonal

$$(1) \qquad \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \qquad (-1). \tag{19}$$

The eigenvalues of these matrices are clearly

$$\lambda = -2, -1, 1, 2. \tag{20}$$

These are also the eigenvalues of $\hat{K}$. Finding the eigenvectors is now straightforward.

$$|-2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \qquad |-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad |2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \tag{21}$$

Solution.c

$$\hat{B}^2 = \frac{1}{2}\left(\hat{I} + |\psi\rangle\langle\psi|\right)\left(\hat{I} + |\psi\rangle\langle\psi|\right) \tag{22}$$

$$= \frac{1}{2}\left(\hat{I} + 3|\psi\rangle\langle\psi|\right) \neq \hat{B} \tag{23}$$

$\hat{B}$ is therefore not a projection operator.

Solution.d

By inspection, we can see that $|\psi\rangle$ is an eigenvector of $\hat{B}$, since both operators in the parenthesis, $\hat{I}$ and $|\psi\rangle\langle\psi|$, yield $|\psi\rangle$ when operating on $|\psi\rangle$. The corresponding eigenvalue is therefore

$$\lambda_{\parallel} = \frac{2}{\sqrt{2}} = \sqrt{2}. \tag{24}$$

Meanwhile, all vectors perpendicular to $|\psi\rangle$ are eigenvectors as well, since the operator $|\psi\rangle\langle\psi|$ yields zero, leaving only $\hat{I}$, for which, all vectors are eigenvectors. Let $E_{\perp}$ be the eigenspace of dimension $n-1 = 3$ spanned by all vectors perpendicular to $|\psi\rangle$. The eigenvalue corresponding to $E_{\perp}$ is

$$\lambda_{\perp} = \frac{1}{\sqrt{2}}. \tag{25}$$

This eigenvalue is 3-fold degenerate.

B.5.6.7

Two psychologists reported on tests in which subjects were given the *prior information*:

I = In a certain, 85% of the taxicabs are blue and 15% are green

and the *data*:

D = A witness (80% reliable) reports that the cab was green

The subjects were then asked to judge the probability that the taxicab was actually blue. What is the correct answer?

Solution

Let B be the event that the cab was actually blue. From Bayes theorem we have.

$$P(B|D \cap I) = P(D|B \cap I)\frac{P(B|I)}{P(D|I)} \tag{26}$$

Because the witness is 80% reliable, there is a 20% likelihood that the witness reports the cab to green if it was actually blue. Therefore

$$P(D|B \cap I) = 0.2. \tag{27}$$

Meanwhile, given the initial conditions, the probability that the car was actually blue is

$$P(B|I) = 0.85. \tag{28}$$

Lastly, we need to find the expectation value of the probability of the witness reporting a green cab given the initial conditions. The witness is 20% likely to report that a blue cab was green and 80% likely to report that a green cab was green. We therefore have

$$P(D|I) = 0.2(0.85) + 0.8(0.15) = 0.29. \tag{29}$$

Plugging everything back into equation 26 gives us

$$P(B|D \cap I) = 0.586. \tag{30}$$

Two cards are drawn at random from a shuffed deck and laid aside without being examined. Then a third card is drawn. Show that the probability that the third card is a spade is the sum of all the possibilies of cards being drawn. The two unexamined cards could be two spades, one spade and one nonspade, one nonspade and one spade, or two nonspades.

$$P(spade_1)P(spade_2)P(spade_3) = (\frac{13}{52})(\frac{12}{51})(\frac{11}{50}) = 0.0129 \tag{1}$$

$$P(spade_1)P(notspade_2)P(spade_3) = (\frac{13}{52})(\frac{39}{51})(\frac{12}{50}) = 0.0459 \tag{2}$$

$$P(notspade_1)P(spade_2)P(spade_3) = (\frac{39}{52})(\frac{13}{51})(\frac{12}{50}) = 0.0459 \tag{3}$$

$$P(notspade_1)P(notspade_2)P(spade_3) = (\frac{39}{52})(\frac{38}{51})(\frac{13}{50}) = 0.1453 \tag{4}$$

The sum of the above probabilites is 0.2500 which is the same as drawing a spade on the first try. The reason these probabilies are the same is because we gain no new information when we did drew the two cards, but did not look at them.

(a) Suppose your friend rolls a pair of dice and, without showing you the result, tells you that sum of the two dice, s = 8. What is your conditional probability distribution for the value of the first die, a?

In order for the two dice to sum up to eight there are five possibilities.

a	b
6	2
5	3
4	4
3	5
2	6

The possible values for a are 2 through 6 and each of the values has a probability of $\frac{1}{5}$ to occur.

(b) Suppose your friend rolls a pair of dice and, without showing you the result, tells you that the product, p = 12. What is your conditional expectation value for s?

In order for the multiply to 12 there are four possible combinations.

a	b
6	2
4	3
3	4
2	6

$6 + 2 = 8$ and $4 + 3 = 7$ therefore the conditional expectation value is...

$$7P(s = 7) + 8P(s = 8) = \frac{15}{2} \tag{1}$$

The probability that a friend and I have different birthdays is given by $prob(\text{me} + \text{friend}) = \frac{364}{365} = 1 - \frac{1}{365}$ since there is one day in the year when the friend could have a birthday in common with the me.

The probability that three people share a birthday is given by

$$prob(3) = prob(2)prob(3 \mid 2) = \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right) \tag{1}$$

where the last term is true since the third person must have a birthday different from the first two birthdays. We can extend this result with induction to get

$$p = prob(n) = prob(2)prob(3 \mid 2)...prob(n \mid n{-}1) = \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right)...\left(1 - \frac{n-1}{365}\right) = p \tag{2}$$

Taking the natural logarithm allows us to separate the product into a sum as

$$ln(p) = ln\left(1 - \frac{1}{365}\right) + ln\left(1 - \frac{2}{365}\right) + ... + ln\left(1 - \frac{n-1}{365}\right) \tag{3}$$

Using the approximation $ln(1+x) \approx x$ for small x we get

$$ln(p) \approx \frac{1}{365}(-1 - 2 - ... - (n-1)) = \frac{-1}{365}\sum_{i=1}^{n-1} i = \frac{-1}{365}\frac{n(n-1)}{2} = \frac{n - n^2}{2 * 365} \tag{4}$$

Now we plug in for p a value of $\frac{1}{2}$ and solve for n, which gives

$$ln\left(\frac{1}{2}\right) = \frac{n - n^2}{2 * 365} \tag{5}$$

Solving the resulting quadratic gives that

$$N = 23 \tag{6}$$

so in a group of 23 people there is a 50% chance that they have no common birthdays.

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For this problem, in ones right pocket there are 3 nickels and 4 dimes and in ones left pocket there are 2 nickels and 1 quarter. You pick a random pocket, and pick a random coin out of that pocket. The coin is a nickel. What is the probability that the nickel came from your right pocket?

To determine this probability we just Bayes' formula. So, we multiply the probability of the coin being a nickel if you picked your right pocket by the probability of you picking your right pocket. Then, divide that by the total probability that the coin is a nickel.

Therefore, we have:

$$\begin{aligned} P(RP|N) &= \frac{P(N|RP) \times P(RP)}{P(N)} \\ &= \frac{\left(\frac{3}{7}\right)\left(\frac{1}{2}\right)}{\frac{1}{2}\left(\frac{3}{7}\right) + \frac{1}{2}\left(\frac{2}{3}\right)} \\ &= \frac{\frac{3}{7}}{\frac{3}{7} + \frac{2}{3}} \\ &= 0.39. \end{aligned}$$

Thus, the probability that the nickel came from the right pocket is 0.39.