A Very Useful Identity with some help...

The operator $U(a) = \exp(ipa/\hbar)$ is a translation operator in space (here we consider only one dimension). To see this we need to prove the identity

$$e^A B a^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + ...$$

Solution

(a) Let

$$B(t) = e^{tA} B e^{-tA} \tag{1}$$

The derivative with respect to time is

$$\begin{aligned} \frac{d}{dt}B(t) &= Ae^{tA}Be^{-tA} + e^{tA}BAe^{-tA} \\ &= e^{tA}ABe^{-tA} + e^{tA}BAe^{-tA} \\ &= e^{tA}[A, B]e^{-tA} \end{aligned} \tag{2}$$

where I have used the fact that A and e^{tA} commute.

(b) If we expand B(t) into a power series, we have

$$B(t) = \sum t^n B_n = \sum t^n \frac{B^n(t)}{n!} \tag{3}$$

The time derivative for any operator that satisfies

$$C(t) = e^{tA} C e^{-tA} \tag{4}$$

is

$$\frac{d}{dt}C(t) = e^{tA}[A, C]e^{-tA}. \tag{5}$$

And so

$$B_n(t) = e^{tA} B_n e^{-tA} = \frac{1}{n}\frac{d}{dt}e^{tA}B_{n-1}e^{-tA} = e^{tA}\frac{[A, B_{n-1}]}{n}e^{-tA}, \tag{6}$$

meaning that

$$B_n = [A, B_{n-1}]/n \tag{7}$$

(c) We want to show by induction that

$$B_n = \frac{1}{n!}C(t) = \underbrace{[A, [A, ...[A,}_{\text{n}} B \underbrace{]...]]}_{\text{n}} \tag{8}$$

It is easy to show that this works for$n = 1$.

$$\frac{d}{dt}B(t) = B_1 = [A, B]/1 \tag{9}$$

Using equation 7, we can then show that it works for $m + 1$ given that it works for m.

$$B_{m+1} = \frac{1}{m+1}[A, B_m] = \frac{1}{m(m+1)}[A, [A, B_{m-1}]] \tag{10}$$

Setting $m + 1 = n$ and expanding out all of the B_i terms, we are left with equation 8.

(d) Setting $t = 1$, we have

$$B(1) = \sum (1)^n B_n = \sum B_n = \sum \frac{1}{n!} \underbrace{[A, [A, ...[A,}_{\text{n}} B \underbrace{]...]]}_{\text{n}}. \tag{11}$$

This completes the proof

(e) We must now show that

$$U(p) = e^{ipa/\hbar} x e^{-ipa/\hbar} = x + a. \tag{12}$$

We just proved that

$$U(p) = x + \frac{ia}{\hbar}[p, x] + \frac{-a^2}{2\hbar}[A, [A, B]] + ... \tag{13}$$

We know from the reading that

$$[p, x] = -i\hbar \tag{14}$$

Because this commutator is constant, all the commutators after $[p, x]$ are zero, and we are left with

$$U(p) = x + \frac{ia}{\hbar}(i\hbar) = x + a \tag{15}$$

We want to prove

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}[\hat{A},\hat{B}]}$$

Given,

$$[\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0$$

First, we start with the function, $f(x) = e^{x\hat{A}}e^{x\hat{B}}$. Next, we take the derivative of $f(x)$.

$$\tfrac{df}{dx} = \hat{A}e^{x\hat{A}}e^{x\hat{B}} + e^{x\hat{A}}\hat{B}e^{x\hat{B}}$$

We can factor out $e^{x\hat{A}}e^{x\hat{B}}$ and multiply $\hat{B}$ by one, $e^{x\hat{A}}e^{-x\hat{A}}$. We then have the differential equation

$$\tfrac{df}{dx} = (\hat{A} + e^{x\hat{A}}\hat{B}e^{-x\hat{A}})f(x)$$

We can substitute the identity found in problem 6.19.11 for $e^{x\hat{A}}\hat{B}e^{-x\hat{A}}$.

$$\tfrac{df}{dx} = (\hat{A} + \hat{B} + [\hat{A},\hat{B}]x + ...)f(x)$$

The higher order terms in the identity will be zero. We can solve the differential equation to get

$$f(x) = e^{x(\hat{A}+\hat{B}+[\hat{A},\hat{B}]x)}$$

$$e^{x\hat{A}}e^{x\hat{B}} = e^{x(\hat{A}+\hat{B}+[\hat{A},\hat{B}]x)}$$

Let's divide both sides by e^x.

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+[\hat{A},\hat{B}]x}$$

Now, we can choose $x = -\frac{1}{2}$ to obtain the equation we were tying to prove.

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+-\frac{1}{2}[\hat{A},\hat{B}]}$$

Maggie Regan
Physics 113
Boccio 6.19.14

In this problem we want to use the equation of motion for the density operator $\hat{\rho}$ to show that a pure state cannot become a nonpure state (and vice versa).

So, we have that $\rho(t) = U\rho_0 U^\dagger$. Then, $(\rho(t))^2 = U\rho_0^2 U^\dagger$. Next, if we take the trace of this, we have $tr((\rho(t))^2) = tr(U\rho_0^2 U^\dagger) = tr(\rho_0^2)$.

Because of this, we know that the state stays at its initial state. Therefore, a pure state cannot change into a nonpure state and vice versa.

Using the given basis, the first step to finding U_{AB} is to find the $2x2$ matrices corresponding to the given operators. They are given by

$$P_0^A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{1}$$

$$P_1^A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2}$$

$$\sigma_x^B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3}$$

Using the rules for tensor products, this gives that

$$U_{AB} = P_0^A \otimes I_B + P_1^A \otimes \sigma_x^B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{4}$$

Now we are asked to find the eigenvectors by inspection. The first two are clearly given by

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{5}$$

$$\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{6}$$

The last two we can conjecture and find to be

$$\vec{v}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \tag{7}$$

$$\vec{v}_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \tag{8}$$

To determine which are entangled we look for superpositions of states. Since the eigenvectors $\vec{v}_3$ and $\vec{v}_4$ are each a superposition of two states, they are both entangled states.

Maggie Regan
Physics 113
Boccio 6.19.19

This problem is a continuation of the same system as in problem 6.17 and 6.18. We want to find a factorizable input state $|\psi_{AB}^{in}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ such that the output state $|\psi_{AB}^{out}\rangle = U_{AB}|\psi_{AB}^{in}\rangle$ is maximally entangled. So, we want to find a factorizable $|\psi_{AB}^{in}\rangle$ such that $Tr[\tilde{\rho}_A^2] = \frac{1}{2}$, where $\tilde{\rho}_A = Tr_B[|\psi_{AB}^{out}\rangle\langle\psi_{AB}^{out}|]$.

Let $|\psi_{AB}^{in}\rangle = (a_0|0_A\rangle + a_1|1_A\rangle)(b_0|0_B\rangle + b_1|1_B\rangle)$. Then, $|\psi_{AB}^{out}\rangle = (a_0|0_A\rangle + a_1|1_A\rangle)(b_0|1_B\rangle + b_1|0_B\rangle)$. Let $\rho = |\psi_{AB}^{out}\rangle\langle\psi_{AB}^{out}|$.

Then,

$$\rho = \begin{bmatrix} a_0^2b_0^2 & a_0a_1b_0^2 & a_0a_1b_0b_1 & a_0^2b_0b_1 \\ a_0a_1b_0^2 & a_1^2b_0^2 & a_1^2b_0b_1 & a_0a_1b_0b_1 \\ a_0a_1b_0b_1 & a_1^2b_0b_1 & a_1^2b_1^2 & a_0a_1b_1^2 \\ a_0^2b_0b_1 & a_0a_1b_0b_1 & a_0a_1b_1^2 & a_0^2b_1^2 \end{bmatrix}.$$

So, $\tilde{\rho_A} = Tr_B[\rho] = \langle 0_B|\rho|0_B\rangle + \langle 1_B|\rho|1_B\rangle = (a_0^2b_0^2 + a_0^2b_1^2)|0_A\rangle\langle 0_A| + 2a_0a_1b_0b_1|0_A\rangle\langle 1_A| + 2a_0a_1b_0b_1|1_A\rangle\langle 0_A| + (a_1^2b_1^2 + a_1^2b_0^2)|1_A\rangle\langle 1_A|$.

Then, $Tr(\tilde{\rho_A}^2) = (a_0^2b_0^2 + a_0^2b_1^2)^2 + 2(2a_0a_1b_0b_1)^2 + (a_1^2b_1^2 + a_1^2b_0^2)^2 = (a_0b_0)^4 + (a_1b_1)^4 + (a_0b_1)^4 + (a_1b_0)^2 + 12(a_0a_1b_0b_1)^2$.

Solving this equation we have values of $a_0 = a_1 = \frac{1}{\sqrt{2}}$, $b_0 = 1$, and $b_1 = 0$. This gives us $Tr[\tilde{\rho}_A^2] = \frac{1}{2}$.

Tensor-Product Bases

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be a paor of two-dimensional Hilbert spaces with given orthonormal bases $\{|0_A\rangle, |1_A\rangle\}$ and $\{|0_B\rangle, |1_B\rangle\}$. Consider the following entangled state in the joint Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$,

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|0_A 1_B\rangle - |1_A 0_B\rangle)$$

where $|0_A 1_B\rangle$ is short-hand notation for $|0_A\rangle \otimes |1_B\rangle$ and so on. Rewrite this state in terms of a new basis $\{|\tilde{0}_A\tilde{0}_B\rangle, |\tilde{0}_A\tilde{1}_B\rangle, |\tilde{1}_A\tilde{0}_B\rangle, |\tilde{1}_A\tilde{1}_B\rangle\}, where.$

$$\begin{aligned}
|\tilde{0}_A\rangle &= \cos\frac{\phi}{2}|0_A\rangle + \sin\frac{\phi}{2}|1_A\rangle \\
|\tilde{1}_A\rangle &= -\sin\frac{\phi}{2}|0_A\rangle + \cos\frac{\phi}{2}|1_A\rangle
\end{aligned}$$

and similarly for $\{|\tilde{0}_B, |\tilde{1}_B\rangle\}$. Again $|\tilde{0}_A\tilde{0}_B = |\tilde{1}_A\rangle \otimes |\tilde{0}_B\rangle$, etc. Is our particular choice of $|\Psi_{AB}\rangle$ special in some way?

Solution

The matrix

$$S_i = \begin{pmatrix} \cos\phi/2 & \sin\phi/2 \\ -\sin\phi/2 & \cos\phi/2 \end{pmatrix} \tag{1}$$

represents the change in basis shown above, so that

$$[\tilde{\psi_A}] = S_A[\psi_A] \qquad \text{and} \qquad [\tilde{\psi_B}] = S_B[\psi_B]. \tag{2}$$

If we let $C = \cos\phi/2$ and $S = \sin\phi/2$, then matrix

$$S_{AB} = S_A \otimes S_B = \begin{pmatrix} C^2 & CS & CS & S^2 \\ -CS & C^2 & -S^2 & CS \\ -CS & -S^2 & C^2 & CS \\ S^2 & -CS & -CS & C^2 \end{pmatrix} \tag{3}$$

represent the same change in basis in the combined Hilbert space. That is

$$[\tilde{\Psi}_{AB}] = S_{AB}[\Psi_{AB}]. \tag{4}$$

Ψ_{AB} can be written in the combined basis as

$$\Psi_{AB} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \tag{5}$$

We can now write Ψ_{AB} in the new basis

$$[\tilde{\Psi}_{AB}] = S_{AB}\Psi_{AB} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}. \tag{6}$$

We can see that Ψ_{AB} is invariant under this coordinate transformation.

(a) $\rho_{AB} = \frac{3}{8}\left|0_A\right\rangle\left\langle 0_A\right| \otimes \frac{1}{2}(\left|0_B\right\rangle + \left|1_B\right\rangle)(\left\langle 0_B\right| + \left\langle 1_B\right|) + \frac{5}{8}(\left|1_A\right\rangle\left\langle 1_A\right|) \otimes \frac{1}{2}(\left|0_B\right\rangle - \left|1_B\right\rangle)(\left\langle 0_B\right| - \left\langle 1_B\right|)$

We can distribute to see that

$$\rho_{AB} = \tfrac{3}{16}(\left|0_A0_B\right\rangle\left\langle 0_A0_B\right| + \left|0_A0_B\right\rangle\left\langle 0_A1_B\right| + \left|0_A1_B\right\rangle\left\langle 0_A0_B\right| + \left|0_A1_B\right\rangle\left\langle 0_A1_B\right|) + \tfrac{5}{16}(\left|1_A0_B\right\rangle\left\langle 1_A0_B\right| - \left|1_A0_B\right\rangle\left\langle 1_A1_B\right| - \left|1_A1_B\right\rangle\left\langle 1_A0_B\right| + \left|1_A1_B\right\rangle\left\langle 1_A1_B\right|)$$

In matrix form this is

$$\frac{3}{16}\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} + \frac{5}{16}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&-1\\0&0&-1&1\end{pmatrix} = \frac{1}{16}\begin{pmatrix}3&3&0&0\\3&3&0&0\\0&0&5&-5\\0&0&-5&5\end{pmatrix}$$

(b) Therefore, we can now compute the matrix representation of

$$(I^A \otimes P_0^B)\rho_{AB}(I^A \otimes P_0^B) + (I^A \otimes P_1^B)\rho_{AB}(I^A \otimes P_1^B)$$

$$I^A \otimes P_0^B = \begin{pmatrix}1&0\\0&1\end{pmatrix} \otimes \begin{pmatrix}1&0\\0&0\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}$$

$$I^A \otimes P_1^B = \begin{pmatrix}1&0\\0&1\end{pmatrix} \otimes \begin{pmatrix}0&0\\0&1\end{pmatrix} = \begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}$$

$$\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix} \frac{1}{16}\begin{bmatrix}3&3&0&0\\3&3&0&0\\0&0&5&-5\\0&0&-5&5\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix} +$$

$$\begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix} \frac{1}{16}\begin{bmatrix}3&3&0&0\\3&3&0&0\\0&0&5&-5\\0&0&-5&5\end{bmatrix}\begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix} = \frac{1}{16}\begin{bmatrix}3&0&0&0\\0&3&0&0\\0&0&5&0\\0&0&0&5\end{bmatrix}$$

(c) The matrix representation of $\tilde{\rho}_A = Tr[\rho_A B]$.

$$\tilde{\rho}_A = \tfrac{1}{16}(6\left|0_A\right\rangle\left\langle 0_A\right| + 10\left|1_A\right\rangle\left\langle 1_A\right|$$

We want to find the commutators of the dilation generator $\hat{D}$ with $\hat{x}$ and $\hat{p}_x$, and then use them to find $\hat{D}$. Let

$$\hat{U}_D = e^{-ic\hat{D}} \tag{1}$$
$$\hat{U}_x = e^{-ia_x\hat{P}_x} \tag{2}$$

where c and a_x are parameters. Next we compute the following product,

$$\hat{U}_x(-\epsilon)\hat{U}_D(-\epsilon)\hat{U}_x(\epsilon)\hat{U}_D(\epsilon) = (x-\epsilon) \to (e^{-\epsilon}(x-\epsilon)) \to (e^{-\epsilon}(x-\epsilon)+\epsilon) \to x-\epsilon+\epsilon e^{\epsilon} \tag{3}$$
$$\approx x-\epsilon+\epsilon(1+\epsilon) = 1+\epsilon^2 = \hat{U}_x(\epsilon^2) \tag{4}$$

so

$$\hat{U}_x(-\epsilon)\hat{U}_D(-\epsilon)\hat{U}_x(\epsilon)\hat{U}_D(\epsilon) = \hat{I}+\epsilon^2[\hat{P}_x,\hat{D}]+h.o.t. = \hat{U}_x(-i\epsilon^2\hat{P}_x) = \hat{I}-i\epsilon^2\hat{P}_x \tag{5}$$

which implies that

$$[\hat{D},\hat{P}_x] = i\hat{P}_x \tag{6}$$

We now take the commutator of both sides with $\hat{x}$ and get

$$i[\hat{P}_x,\hat{x}] = [[\hat{D},\hat{P}_x],\hat{x}] = [[\hat{P}_x,\hat{x}],\hat{D}]+[[\hat{x},\hat{D}],\hat{P}_x] = [-i\hat{I},\hat{D}]+[[\hat{x},\hat{D}],\hat{P}_x] \tag{7}$$
$$= [[\hat{x},\hat{D}],\hat{P}_x] = i(-i\hat{I}) = \hat{I} \tag{8}$$

where the Jacobi identity was applied to expand the commutator, and the commutator of anything with $\hat{I}$ is zero. This expression implies that

$$[\hat{D},\hat{x}] = i\hat{x} \tag{9}$$

Together with

$$[\hat{D},\hat{P}_x] = i\hat{P}_x \tag{10}$$

they imply that $\hat{D}$ must have a form like

$$\hat{D} = \hat{x}\hat{P}_x \tag{11}$$

But since $\hat{D}$ is Hermitian, it must be self-adjoint so this requires the form

$$\hat{D} = \frac{1}{2}(\hat{x}\hat{P}_x + \hat{P}_x\hat{x}) \tag{12}$$