Given an infinite potential well from $0 < x < L$ with a delta function potential $\lambda\delta(x - \frac{L}{2})$ at $\frac{L}{2}$. We know that the wavefunction will be in the form

$$\Psi = \begin{cases} A\sin(kx), & \text{if } 0 < x < \frac{L}{2} \\ B\sin(k(x-L)), & \text{if } \frac{L}{2} < x < L \end{cases}$$

Since the function needs to be zero at $x = 0$ and $x = L$ we only have the sine terms. Next, we can use the boundary conditions to solve for A and C. First, $A\sin(kx)$ and $B\sin(k(x-L))$ need to be equal at $x = \frac{L}{2}$.

$$A\sin(k\tfrac{L}{2}) = B\sin(k(-\tfrac{L}{2}))$$

$$A\sin(k\tfrac{L}{2}) = -B\sin(k\tfrac{L}{2})$$

$$A = -B$$

Let's rename $A\sin(kx) = \Psi_1$ and $B\sin(k(x-L)) = \Psi_2$.The second condition is $\frac{\partial\Psi_2}{\partial x} - \frac{\partial\Psi_1}{\partial x} = E\Psi_1$. In this case, $E = \frac{2m\lambda}{\hbar^2}$.

$$Bk\cos(k(-\tfrac{L}{2})) - Ak\cos(k\tfrac{L}{2}) = \tfrac{2m\lambda}{\hbar^2}A\sin(k\tfrac{L}{2})$$

$$Bk\cos(k\tfrac{L}{2}) - Ak\cos(k\tfrac{L}{2}) = \tfrac{2m\lambda}{\hbar^2}A\sin(k\tfrac{L}{2})$$

$$\tfrac{k(B-A)}{A}\tfrac{\hbar^2}{2m\lambda} = \tan(k\tfrac{L}{2})$$

$$-\tfrac{k\hbar^2}{m\lambda} = \tan(k\tfrac{L}{2})$$

This is a transcendental equation for the energy eigenvalues.

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In this problem we have an atomic model where we are considering a particle moving under the influence of the one-dimensional potential given by

$$V(x) = \begin{cases} -V_0\delta(x) & x > -d \\ \infty & x < -d \end{cases}$$

We are able to guess at a general solution of:

$$\psi(x) = \begin{cases} 0 & x < -d \\ A_1 e^{kx} + A_2 e^{-kx} & -d < x < 0 \\ B_1 e^{kx} + B_2 e^{-kx} & 0 < x < \infty \end{cases}$$

Right away we know that $B_1 = 0$ because we do not want $\psi(x)$ to blow up at infinity. Therefore,

$$\psi(x) = \begin{cases} 0 & x < -d \\ A_1 e^{kx} + A_2 e^{-kx} & -d < x < 0 \\ B_2 e^{-kx} & 0 < x < \infty \end{cases}$$

Next, we check all of the end points of the intervals. We know that there are not any discontinuities, so setting these equal will give us relationships between the constants A_1, A_2 and B_2.

So, $\psi(-d) = 0 = A_1 e^{-kd} + A_2 e^{kd}$. Thus, $A_1 = -A_2 e^{2kd}$.

Then, we look at when $x = 0$. So, $A_2(1 - e^{2kd}) = B_2$.

Next, we use these relations and look at the discontinuities of the derivatives. So, we have that after algebra, and solving for k: $k = \frac{mv_0}{\hbar^2}(1 - e^{-2kd})$. This is the transcendental equation that we wanted.

For part (b), we want to find an approxiamation for the modification of the bound-state energy caused by the wall when it is far away. When the wall is infinitely far away, we have that $k^{(0)} = \frac{mV_0}{\hbar^2}$. Then, if the wall is very far away, but not infinitely far away, we have that $k^{(1)} = \frac{mV_0}{\hbar^2}(1 - e^{-2k^{(0)}d}) = \frac{mV_0}{\hbar^2} - \frac{mV_0}{\hbar^2} e^{\frac{2mV_0}{d\hbar^2}} = k^{(0)} - \frac{mV_0}{\hbar^2} e^{\frac{2mV_0}{d\hbar^2}}$. Therefore, the presence of the wall creates the presence of the factor above that decreases the value of k.

Then, in part (c) we are asked to find the exact condition on V_0 and d for the existence of at least one bound state. We do this by look at the derivative of k. The conditions are found by looking at the point where the derivative at $k = 0$ is equal to 1. Thus, we have that $\frac{2mV_0 d}{\hbar^2} e^{-2kd} = \frac{2mV_0 d}{\hbar^2} e^{-2(0)d} = \frac{2mV_0 d}{\hbar^2} = 1$. Therefore, the condition is $V_0 d = \frac{\hbar^2}{2m}$. This condition will lead to at least one intersection.

a

We are given that the electron is confined to $x > 0$ and has potential energy

$$V(x) = -\frac{e^2}{4x}$$

and we know that the energy of the electron is negative, so we can write the Schrodinger equation as

$$-\frac{\hbar^2}{2m}\phi'' = (\frac{e^2}{4x} - |E|)\phi$$

In the limit $x \to \infty$, the potential term disappears and we get

$$\phi'' = \frac{2m\,|E|}{\hbar^2}\phi$$

This gives a solution of the form

$$\phi(x) = Ae^{kx} + Be^{-kx}$$

where

$$k = \frac{\sqrt{2m\,|E|}}{\hbar}$$

But we know that as $x \to \infty$, $e^{kx} \to \infty$ so we must have that $A = 0$. Then

$$\phi(x) = Be^{-kx}$$

b

As $x \to 0^+$, $V(x) \to \infty$. So,

$$V(0) = \infty$$

which implies that

$$\phi(0) = 0$$

c

We guess a solution of the form

$$\phi(x) = B(x)e^{-kx}$$

where $B(x)$ is to be determined, such that it satisfies

$$\phi(0) = B(0) = 0$$

Now we take derivatives of $\phi(x)$ which we will plug back into Schrodinger's equation to solve for $B(x)$.

$$\phi'(x) = e^{-kx}(B' - kB)$$

$$\phi''(x) = e^{-kx}(-k(B' - kB) + B'' - kB') = e^{-kx}(k^2B - 2kB' + B'')$$

Substituting back in gives

$$-\frac{\hbar^2}{2m}e^{-kx}(k^2B - 2kB' + B'') = (\frac{e^2}{4x} - |E|)B$$

We can guess $B(x) = Cx$ is a solution for some constant C, which implies that

$$\phi(x) = Cxe^{-kx}$$

Plugging in to the original Schrodinger equation and dividing through by C gives

$$-\frac{\hbar^2}{2m}e^{-kx}(k^2x - 2k) = (\frac{e^2}{4x} - |E|)x$$

so

$$-\frac{\hbar^2}{2m}k^2 = -|E|$$
$$-\frac{\hbar^2}{2m}(-2k) = \frac{e^2}{4}$$

The first equation gives us the relationship between E and k which we found above. The second gives that

$$k = \frac{me^2}{4\hbar^2} = \frac{1}{4a_o}$$

where a_o is the Bohr radius. Now we must normalize the wavefunction $\phi(x)$ by setting $\int_0^\infty |\phi^2(x)|\,dx = 1$. This gives

$$\int_0^\infty \left|\phi^2(x)\right| dx = C^2 \int_0^\infty x^2 e^{-2kx} dx = C^2 \frac{1}{4k^3} = 1$$

Therefore,

$$C^2 = \frac{1}{16a_o^3}$$

and

$$C = \pm\frac{1}{4\sqrt{a_o^3}}$$

which gives

$$\phi(x) = \pm\frac{1}{4\sqrt{a_o^3}} x e^{-kx}$$

d

The ground state energy can be found from k using the formula above:

$$|E| = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2(\frac{1}{4a_o})^2}{2m} = \frac{\hbar^2}{32ma_o^2} = \frac{1}{8}\frac{e^2}{a_o}$$

so

$$E = -\frac{1}{8}\frac{e^2}{a_o}$$

e

We compute

$$\langle \hat{x} \rangle = \langle \phi(x) | \, \hat{x} \, | \phi(x) \rangle = \frac{1}{16a_o^3} \int_0^\infty x^3 e^{-2kx} dx = \frac{1}{16a_o^3} \frac{3}{2k} \int_0^\infty x^2 e^{-2kx} dx$$
$$= \frac{1}{16a_o^3} \frac{3}{2k} \frac{1}{4k^3} = \frac{1}{16a_o^3} \frac{3}{8k^4} = 6a_o$$

so

$$\langle \hat{x} \rangle = 6a_o$$

We want to prove

$$\langle 0| e^{ik\hat{x}} |0\rangle = exp[-k^2 \langle 0| \hat{x}^2 |0\rangle /2]$$

We are given that $\hat{x}$ is the position operator so $\hat{x} = \sqrt{\frac{\hbar}{2m\omega_0}}(\hat{a} + \hat{a}^+)$.

$$\langle 0| e^{ik\hat{x}} |0\rangle = \langle 0| e^{ik\sqrt{\frac{\hbar}{2m\omega_0}}(\hat{a}+\hat{a}^+)} |0\rangle$$

We also remember from a few weeks ago that $e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}[\hat{A},\hat{B}]}$. So,

$$\langle 0| e^{ik\hat{x}} |0\rangle = \langle 0| e^{ik\sqrt{\frac{\hbar}{2m\omega_0}}\hat{a}^+} e^{ik\sqrt{\frac{\hbar}{2m\omega_0}}\hat{a}} e^{\frac{1}{2}k^2 \frac{\hbar}{2m\omega_0}[\hat{a}^+,\hat{a}]} |0\rangle$$

$$= e^{-\frac{1}{4}\frac{\hbar}{m\omega_0}k^2} \langle 0| e^{ik\sqrt{\frac{\hbar}{2m\omega_0}}\hat{a}^+} e^{ik\sqrt{\frac{\hbar}{2m\omega_0}}\hat{a}} |0\rangle$$

$$= e^{-\frac{1}{4}\frac{\hbar}{m\omega_0}k^2} \langle 0| e^{ik\sqrt{\frac{\hbar}{2m\omega_0}}\hat{a}^+} e^{ik\sqrt{\frac{\hbar}{2m\omega_0}}\hat{a}} |0\rangle$$

We can Taylor expand $e^{ik\sqrt{\frac{\hbar}{2m\omega_0}}\hat{a}}$.

$$= e^{-\frac{1}{4}\frac{\hbar}{m\omega_0}k^2} \langle 0| e^{ik\sqrt{\frac{\hbar}{2m\omega_0}}\hat{a}^+} (\hat{I} + i\sqrt{\frac{k\hat{a}\hbar}{2m\omega_0}} + ...) |0\rangle$$

We only care about the first term, $\hat{I}$, and $\hat{I}|0\rangle = |0\rangle$. We also know that the Hermitian conjurgate will equal the same thing, so

$$\langle 0| e^{ik\hat{x}} |0\rangle = e^{-\frac{1}{4}\frac{\hbar}{m\omega_0}k^2} \langle 0| |0\rangle$$

$$\langle 0| e^{ik\hat{x}} |0\rangle = e^{-\frac{1}{4}\frac{\hbar}{m\omega_0}k^2}$$

We also know that $\langle 0| \hat{x}^2 |0\rangle = \frac{\hbar}{2m\omega_0} \langle 0| (\hat{a}^2 + \hat{a}^{+2} + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a} |0\rangle = \frac{\hbar}{2m\omega_0}$. Therefore,

$$\langle 0| e^{ik\hat{x}} |0\rangle = e^{-\frac{1}{4}\frac{\hbar}{m\omega_0}k^2} = e^{-\frac{k^2}{2}\langle 0|\hat{x}^2|0\rangle}$$

We are given that the new state vector is

$$|\phi\rangle = e^{-ip_o\hat{x}/\hbar}|0\rangle$$

The probability that the system is still in the ground state $|0\rangle$ is given by

$$Prob = |\langle\phi|\,0\rangle|^2 = \left|\langle 0|\,e^{ip_o\hat{x}/\hbar}\,|0\rangle\right|^2$$

We can rewrite $\hat{x}$ in the basis of raising and lowering operators as

$$\hat{x} = x_o(\hat{a}^\dagger + \hat{a})$$

where

$$x_o = \sqrt{\frac{\hbar}{2m\omega_o}}$$

so

$$Prob = \left|\langle 0|\,e^{ip_o\hat{x}/\hbar}\,|0\rangle\right|^2 = \left|\langle 0|\,e^{ip_o x_o(\hat{a}^\dagger+\hat{a})/\hbar}\,|0\rangle\right|^2 = \left|\langle 0|\,e^{ip_o x_o\hat{a}^\dagger/\hbar}e^{ip_o x_o\hat{a}/\hbar}e^{\frac{1}{2}p_o^2x_o^2[\hat{a}^\dagger,\hat{a}]/\hbar}\,|0\rangle\right|^2$$

using a previous identity. Then since

$$[\hat{a}^\dagger,\hat{a}] = -1$$

and

$$e^{ip_o x_o\hat{a}/\hbar}\,|0\rangle = (\hat{I} + \frac{1}{\hbar}ip_o x_o\hat{a} + ...)\,|0\rangle = |0\rangle$$

and similarly

$$\langle 0|\,e^{ip_o x_o\hat{a}^\dagger/\hbar} = \langle 0|\,(\hat{I} + \frac{1}{\hbar}ip_o x_o\hat{a}^\dagger + ...) = \langle 0|$$

we have that

$$Prob = \left| \langle 0| \, e^{ip_o x_o \hat{a}^\dagger/\hbar} e^{ip_o x_o \hat{a}/\hbar} e^{\frac{1}{2}p_o^2 x_o^2 [\hat{a}^\dagger, \hat{a}]/\hbar} \, |0\rangle \right|^2 = \left| (\langle 0| \, e^{ip_o x_o \hat{a}^\dagger/\hbar}) e^{-\frac{1}{2}p_o^2 x_o^2} (e^{ip_o x_o \hat{a}/\hbar} \, |0\rangle) \right|^2$$

$$= \left| \langle 0| \, e^{-\frac{1}{2}p_o^2 x_o^2} \, |0\rangle \right|^2 = e^{-p_o^2 x_o^2}$$

So the probability that the system stays in its ground state is

$$Prob = e^{-p_o^2 x_o^2}$$

Each coherent state has a complex label z and is given by $|z\rangle = e^{z\hat{a}^+}\langle 0|$.

(a) We want to show $\hat{a}|z\rangle = \hat{a}e^{z\hat{a}^+}|0\rangle$.

First, we can Taylor expand $e^{z\hat{a}^+}$.

$$\hat{a}e^{z\hat{a}^+}|0\rangle = \hat{a}\sum_{n=0}^{\infty}\tfrac{z^n}{n!}(\hat{a}^+)^n|0\rangle$$

$$= \hat{a}\sum_{n=0}^{\infty}\tfrac{z^n}{n!}\sqrt{n!}\,|n\rangle$$

$$= \sum_{n=1}^{\infty}\tfrac{z^n}{(n-1)!}\sqrt{(n-1)!}\,|n-1\rangle$$

To have all of the n terms inside the summation to be $n-1$ we can factor a z out of the equation.

$$= z\sum_{n=1}^{\infty}\tfrac{z^{n-1}}{(n-1)!}\sqrt{(n-1)!}\,|n-1\rangle$$

$$= z\sum_{n=1}^{\infty}\tfrac{z^{n-1}}{(n-1)!}(\hat{a}^+)^{n-1}|n-1\rangle$$

The summation is just another Taylor expansion of $e^{z\hat{a}^+}$, therefore

$$\hat{a}e^{z\hat{a}^+}|0\rangle = z|z\rangle$$

(b) We want to show $\langle z_1|\, z_2\rangle = e^{z_1^* z_2}$.

First, we know that

$$\langle z_1|\, z_2\rangle = \langle 0|\, e^{z_1^*\hat{a}}e^{z_2\hat{a}^+}|0\rangle$$

$$= \langle 0|\, e^{z_1^* z_2\hat{a}\hat{a}^+}|0\rangle$$

We can rotate the order of $\hat{a}$ and $\hat{a}^+$ because $\hat{a}\hat{a}^+ = \hat{a}^+\hat{a} + 1$.

$$\langle z_1|\, z_2\rangle = \langle 0|\, e^{z_1^* z_2(\hat{a}^+\hat{a}+1)}|0\rangle$$

We can the distribute $z_1^* z_2$.

$$\langle z_1|\, z_2\rangle = \langle 0|\, e^{z_1^* z_2\hat{a}^+\hat{a}}e^{z_1^* z_2}|0\rangle$$

Similar to what we showed in part a, $e^{z_1^* z_2\hat{a}^+\hat{a}} = e^{z_1^* z_1\hat{a}} = 1$.

$$\langle z_1|\, z_2\rangle = \langle 0|\, e^{z_1^* z_2}|0\rangle = e^{z_1^* z_2}.$$

(c) We want to show that the completeness relation takes the form

$$\hat{I} = \sum_n |n\rangle \langle n| = \int \frac{dxdy}{\pi} |z\rangle \langle z| e^{-z^* z}$$

Where $|n\rangle$ is a standard harmonic oscillator energy eigenstate, $\hat{I}$ is the identity operator, $z = x + iy$, and the integration is taken over the whole $x - y$ plane.

First, we have $z = x + iy$ which in polar coordinates is $z = re^{i\theta}$. From part a, we know that

$$|z\rangle \langle z| = e^{z\hat{a}^+} |0\rangle \langle 0| e^{z^* \hat{a}^+}$$

$$= \sum_n \frac{z^n}{n!} (\hat{a}^+)^n |0\rangle \langle 0| \sum_n \frac{(z^*)^n}{n!} (\hat{a}^+)^n$$

$$= \sum_{n,m} \frac{r^{n+m} e^{i(n+m)\theta}}{n!m!} (\hat{a}^+)^n |0\rangle \langle 0| (\hat{a})^m$$

Now we can plug $|z\rangle \langle z|$ back into $\int \frac{dxdy}{\pi} |z\rangle \langle z| e^{-z^* z}$.

$$\int \frac{dxdy}{\pi} (\sum_{n,m} \frac{r^{n+m} e^{i(n+m)\theta}}{n!m!} (\hat{a}^+)^n |0\rangle \langle 0| (\hat{a})^m) e^{-z^* z} =$$

$$\frac{1}{\pi} \int_0^\infty \int_0^{2\pi} e^{-r^2} r (\sum_{n,m} \frac{r^{n+m} e^{i(n+m)\theta}}{n!m!} (\hat{a}^+)^n) d\theta dr =$$

$$2 \int_0^\infty e^{-r^2} r (\sum_n \frac{r^{2n}}{n!^2} (\hat{a}^+)^n) |0\rangle \langle 0| (\hat{a})^n dr =$$

$$2 \int_0^\infty e^{-r^2} r (\sum_n \frac{r^{2n}}{n!^2} \sqrt{n!} |n\rangle \langle n| \sqrt{n!} dr =$$

$$2 (\sum_n \frac{1}{n!} |n\rangle \langle n| \int_0^\infty r e^{-r^2} r^{2n} dr =$$

If we let $r^2 = x$, then $\int_0^\infty r e^{-r^2} r^{2n} dr = \frac{1}{2} \Gamma(1 + n)$.

$$2 (\sum_n \frac{1}{n!} |n\rangle \langle n| \frac{1}{2} \Gamma(1_n) =$$

$$\sum_n |n\rangle \langle n| = \hat{I}$$

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In this problem we want to obtain the eigenstates of the Hamiltonian $\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a} + V\hat{a} + V^*\hat{a}^\dagger$. We want to do this for a complex V using coherent states.

We start by letting $\hat{a} = \hat{b} + \alpha$ and $\hat{a}^\dagger = \hat{b}^\dagger + \alpha^*$. Then,

$$\begin{aligned}\hat{H} &= \hbar\omega(\hat{b}^\dagger + \alpha^*)(\hat{b} + \alpha) + V(\hat{b} + \alpha) + V^*(\hat{b}^\dagger + \alpha^*) \\ &= \hbar\omega\hat{b}^\dagger\hat{b} + \hbar\omega(\alpha^*\hat{b} + \alpha\hat{b}^\dagger) + \hbar\omega\alpha^*\alpha + V\hat{b} + V\alpha + V^*\hat{b}^\dagger + V^*\alpha^*\end{aligned}$$

Then, if we let $\alpha = \frac{-V^*}{\hbar\omega}$, this gives $\alpha^* = \frac{-V}{\hbar\omega}$. Plugging these values in we have:

$$\begin{aligned}\hat{H} &= \hbar\omega\hat{b}^\dagger\hat{b} + (-V\hat{b} - V^*\hat{b}^\dagger) + \tfrac{VV^*}{\hbar\omega} + V\hat{b} - \tfrac{VV^*}{\hbar\omega} + V^*\hat{b}^\dagger - \tfrac{V^*V}{\hbar\omega} \\ &= \hbar\omega\hat{b}^\dagger\hat{b} - \tfrac{1}{\hbar\omega}|V|^2.\end{aligned}$$

This yields the equation for a harmonic oscillator as in the text, but it is just shifted by a constant. So, as before we have that $\hat{a}^\dagger\hat{a}|N\rangle = N|N\rangle$. Thus, $\hat{H} = (\hbar\omega\hat{b}^\dagger\hat{b} - \frac{1}{\hbar\omega}|V|^2)|N\rangle = \hbar\omega N - \frac{1}{\hbar\omega}|V|^2|N\rangle = E_N|N\rangle$. Therefore, we obtain the same energy values, only they are displace by a constant, $\frac{-1}{\hbar\omega}|V|^2$.

Now, we have that the new groud state is given by:

$$a|N=0\rangle = (b+\alpha)|N=0\rangle \; b|N=0\rangle = -\frac{V^*}{\hbar\omega}|N=0\rangle$$

Therefore, the new ground state is just an eigenstate of $\hat{a}$ with an eigenvalue of $\alpha = -\frac{V^*}{\hbar\omega}$. This new ground state is the coherent state of this Hamiltonian operator.

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Boccio 8.15.21.

In this problem we have a 1-dimensional harmonic oscillator that at the time $t = 0$, is in the state

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle),$$

where $|n\rangle$ is the n^{th} eigenstate. We want to find the expectation value of position and energy at time t.

So, we have that $\hat{H} = \hbar\omega(n + \frac{1}{2})|n\rangle$. Then, $|\psi(t)\rangle = \frac{1}{\sqrt{3}} e^{\frac{-i\omega t}{2}}(|0\rangle + e^{-i\omega t}|1\rangle + e^{-2i\omega t}|2\rangle)$.

$$\begin{aligned}
\langle x \rangle &= \langle \psi(t)|x|\psi(t)\rangle \\
&= (\frac{1}{\sqrt{3}} e^{\frac{i\omega t}{2}}[\langle 0| + e^{i\omega t}\langle 1| + e^{2i\omega t}\langle 2|])x(\frac{1}{\sqrt{3}} e^{\frac{-i\omega t}{2}}[|0\rangle + e^{-i\omega t}|1\rangle + e^{-2i\omega t}|2\rangle]) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left(\frac{1}{3}\right)([\langle 0| + e^{i\omega t}\langle 1| + e^{2i\omega t}\langle 2|])(a + a^\dagger)([|0\rangle + e^{-i\omega t}|1\rangle + e^{-2i\omega t}|2\rangle]) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left(\frac{1}{3}\right)([\langle 0| + e^{i\omega t}\langle 1| + e^{2i\omega t}\langle 2|])([|1\rangle + e^{-i\omega t}|0\rangle + \sqrt{2}e^{-i\omega t}|2\rangle + \sqrt{2}e^{-2i\omega t}|1\rangle]) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left(\frac{1}{3}\right)(e^{-i\omega t} + e^{i\omega t} + \sqrt{2}e^{-i\omega t} + \sqrt{2}e^{i\omega t}) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left(\frac{1}{3}\right)(\cos(\omega t) - i\sin(\omega t) + \cos(\omega t) + i\sin(\omega t) \\
&+ \sqrt{2}\cos(\omega t) - i\sqrt{2}\sin(\omega t) + \sqrt{2}\cos(\omega t) + i\sqrt{2}\sin(\omega t)) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left(\frac{1}{3}\right)(2\cos(\omega t) + 2\sqrt{2}\cos(\omega t)) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left(\frac{2}{3}\right)\cos(\omega t)(1 + \sqrt{2}).
\end{aligned}$$