A brief introduction to f(R) gravity

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Here I attempt to make a brief introduction to what f(R) gravities are, and how they work, trying to avoid (as long as it is possible) complicated mathematical treatments. The objective, is to introduce the reader to the different types of f(R) gravity and its manipulation, keeping in mind that the reader is an undergraduate student that, even when he or she has some knowledge of General Relativity and Cosmology, is far from being an expert in the area.

I. INTRODUCTION

Recent observations show that the universe is expanding at an accelerating rate. This, of course, introduces the problem that GR does not predict this to happen, at least not in a Universe composed only by matter. To attack this problem, two general ways have been taken: Introducing a new type of energy (such as the cosmological constant Λ, dark energy) or modifying the theory of gravitation (such as MOND). Here, we review f(R) gravity, a modification to GR that, even when it's initial motivation wasn't solving the problem of the accelerating Universe (the first f(R) gravities were prior to this observation), it seems that it might help us.

II. ACTION AND LAGRANGIAN CONCEPTS ON GR

The concept of action, as well as the concept of lagrangian, is very familiar in the context of classical mechanics, but not as much in the context of GR. However, both are very important concepts in further discussions, so we must introduce them in this section.

A. The action in classical mechanics

Let us first recall some things about the classical action that will be important in some later analogies. The action in classical mechanics is defined as:

$$S = \int L(q, \dot{q})dt \tag{2.1}$$

Where L is the lagrangian of the system wrote in the form:

$$L = L(q, \dot{q}) \tag{2.2}$$

The action it's a quantity that, using Hamilton's principle (which you can review on [1]), states the "trajectory" that a body will follow. Hamilton's principle states that a body will follow the trajectory which satisfies:

$$\delta S = 0 \tag{2.3}$$

It can be demonstrated (a careful derivation may be found in [1]):

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \Rightarrow \delta S = 0 \tag{2.4}$$

Which means that, if the left side of relation (2.4) is satisfied, then so is eq. (2.3). This comes in handy in classical mechanics, giving us the chance to use a much more practical mathematical treatment of the problem, avoiding variational calculus. However we will see later, that this is not an option in GR.

B. Lagrangian and action in GR

The lagrangian's definition it's not very different in GR than in classical mechanics. The main difference between classical and relativistic lagrangians, lies in the fact that, in GR, we have a curved space-time, and so, we must associate a lagrangian to the vacuum space. Note that this is a completely new idea of what we know from the lagrangian to be in classical mechanics. The derivation of the lagrangian associated to the vaccum is complicated and requires a lot of mathematical work (the fact that we are now working in fields theory makes the whole deal much more complicated), in which we are not interested here, so, we are going to state some things that may give us a intuition of why the lagrangian is what it is. A detailed derivation can be found on [2].

We are trying to associate a lagrangian to the vacuum and we know that our curved space-time is characterized by the metric tensor $g^{\alpha\beta}$, so the lagrangian must depend on nothing else that the metric tensor and it's derivates. It's also very natural to think that we must introduce information related to the curvature of the space-time, so we also must include the Riemman and Ricci tensors in our lagrangian. Since the lagrangian is a scalar quantity, we must think of a scalar quantity that can be derivated from the metric tensor and contains information about the curvature of the space-time. So it's completely natural to think that the Ricci scalar is the quantity we are looking for (at least, at first order). This brings two problems to our analysis. First, the Ricci scalar depends on second order derivates of the metric tensor (note that the first order derivates of the metric tensor are zero) and so, we will not be able to write the lagrangian in the form of eq. (2.2), which means we won't be able either to use the relation (2.4). Second, we must keep in mind that we are going to integrate the lagrangian, but the differential that we must consider to do this, is not invariant, so we must add a term to the lagrangian that helps us with this. This is as far as we can go with non mathematical arguments. But even then is intuitive to write the lagrangian of the vacuum as:

$$\delta L = \sqrt{-g}R \tag{2.5}$$

Where R is the Ricci scalar and $\sqrt{-g}$ is the square root of the determinant of the metric tensor (this is the term we have added in order to be able to integrate the lagrangian). With this, we can write the action.
It might call your attention that, we assumed the lagrangian to be proportional to R (not considering possible dependence on powers or functions of R), and the argument for this is simplicity. We have built the simplest lagrangian that contents all the information necessary about the space-time.
In GR, the main difference lies in the fact that we are working in a four dimension manifold, so the action has the form:

$$S = \frac{1}{2\kappa}\int L d^4x \tag{2.6}$$

Where we are integrating the four dimensions and:

$$\kappa = \frac{8\pi G}{c^2} \tag{2.7}$$

The factor $\frac{1}{2\kappa}$ appears as a normalization that will throw the right terms in Einstein and Friedman equations.
Using eq. (2.3) (with variation respect to the metric tensor) from this lagrangian, will give us Einstein's field equations for the vacuum. If we want to obtain Einstein's equations in the presence of matter, all we need to do is adding a term to the action, associated with matter, i.e:

$$S = \frac{1}{2\kappa}\int \sqrt{-g}R d^4x + S_M \tag{2.8}$$

This is known as the Einstein-Hilbert action, where:

$$S_M = \int \sqrt{-g}L_M d^4x d^4x \tag{2.9}$$

And then define the stress-energy tensor as:

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta L_M}{\delta g^{\mu\nu}} = -2\frac{\delta L_M}{\delta g^{\mu\nu}} + g_{\mu\nu}L_M \tag{2.10}$$

Now that we are more familiarized with the concepts of action and lagrangian in GR (and therefore, cosmology) we may attempt to understand how f(R) gravities work.

III. F(R) GRAVITIES

f(R) gravities are all about modifying the Einstein-Hilbert action and taking it to higher orders in R so the lagrangian will take the form:

$$L = \sqrt{-g}f(R) \tag{3.1}$$

f(R) gravities were not born to explain the accelerating universe, in fact, they first appeared as questioning of the simplicity argument we talked about previously. More than one person, was not convinced that the lagrangian should have the form of eq. (2.5), just because it is the simplest form that involves enough information about the curvature. In this spirit, some people introduced new lagrangians in the form of eq. (3.1).

You might wonder why we aren't introducing terms like $R^{\alpha\beta}R_{\alpha\beta}$, and the answer is very simple: because we are trying to maintain simplicity. f(R) gravities are useful to help us understand this problem from a mathematical point of view, so we can latter associate a physical model that fits. Including terms with Ricci's tensor, make this process extremely complicated and so f(R) gravities lose their utility. This might seem a little ridiculous, since the motivation for f(R) gravity was that people didn't rely in the argument of simplicity, but that doesn't mean that we are going to take things at the most complicated level. Here we are trying sort of an equilibrium: we are not reducing things to the simplest case of all, but we are neither taking them to the hardest case.

There are three types of f(R) gravities, and here we review them.

A. Metric f(R) gravity

In this formalism we write the action as:

$$S = \frac{1}{2\kappa}\int \sqrt{-g}f(R)d^4x + S_M \tag{3.2}$$

Variation with respect to the inverse metric tensor yields the field equation:

$$f'(R)R_{\alpha\beta} - \frac{f(R)}{2}g_{\alpha\beta} = \nabla_\alpha\nabla_\beta f'(R) - g_{\alpha\beta}\nabla^\mu\nabla_\mu f'(R) + \kappa T_{\alpha\beta} \quad (3.3)$$

The process leading to eq. (3.3) is highly non trivial, but since it's not in our best interest to show a detailed mathematical process, we will give some hints you can follow to get there. In order to achieve a better understanding of this, you may review the derivation of Einstein's field equations from the Einstein-Hilbert action in [3]. What we have done here is very similar.

In order to get to eq. (3.3) from eq. (3.2), you should recall what you (hopefully) learned from the derivation of Einstein's field equations. This is:

$$\delta(f(R)) = \frac{\partial f(R)}{\partial R}\delta R = f'(R)\delta R \quad (3.4)$$

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \quad (3.5)$$

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}(\nabla_\rho\delta\Gamma^\rho_{\nu\mu} - \nabla_\nu\delta\Gamma^\rho_{\rho\mu}) \quad (3.6)$$

Note that, in this case (not as in the derivation using Einstein-Hilbert action), the second term won't vanish since this time, we can't write as a total derivative and use Stoke's theorem (there will be a partial derivate of f(R) that will bother us). Then you will need to express the second term on eq. (3.6) in terms of variations respect to the inverse metric tensor, to do so, resort to:

$$\delta\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\lambda\alpha}(\nabla_\mu\delta g_{\alpha\nu} + \nabla_\nu\delta g_{\alpha\mu} - \nabla_\alpha\delta g_{\mu\nu}) \quad (3.7)$$

After which you should obtain:

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g_{\mu\nu}\nabla^\alpha\nabla_\alpha\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu} \quad (3.8)$$

Then recall eq. (2.3) and, integrating by parts, you should obtain eq. (3.3).

B. Palatini f(R) gravity

In Palatini f(R) gravity we assume that both, the metric tensor ($g^{\alpha\beta}$) and the Levi-Civita connection ($\Gamma^\rho_{\mu\nu}$) are independent variables, and so we have two Ricci scalars. We shall denote the Ricci scalar obtained from the independent Levi-Civita connection as $\tilde{R} = g^{\mu\nu}\tilde{R}_{\mu\nu}$, where $\tilde{R}_{\mu\nu}$ is the Ricci tensor constructed with the independent Levi-Civita connection. The action is written as:

$$S = \frac{1}{2\kappa}\int\sqrt{-g}f(\tilde{R})d^4x + S_M \quad (3.9)$$

Where we have assumed that the matter part of the action does not depends of the independent Levi-Civita connection, but, what does this means? It means that we will consider a Levi-Civita connection (yet unknown) that cannot be constructed form the metric tensor, note that this is the reason that the Ricci scalar it's not the same as in metric f(R) gravity. Then, how we define the independent Levi-Civita connection? In the mathematical process leading to the field equations (that we won't cover here for being much more complicated than the process followed in the case of metric f(R) gravity) it comes up a term that suggest a definition. We define a conformal metric as:

$$h_{\mu\nu} = f'(\tilde{R})g_{\mu\nu} \quad (3.10)$$

From here, similarly to what we do with the common metric tensor we define:

$$\Gamma^\lambda_{\mu\nu} = h^{\lambda\sigma}(h_{\nu\sigma,\mu} + h_{\mu\sigma,\nu} - h_{\mu\nu,\sigma}) \quad (3.11)$$

After quite some work, you get to the field equations:

$$f'(\tilde{R})\tilde{R}_{\alpha\beta} - \frac{f(\tilde{R})}{2}g_{\alpha\beta} = \kappa T_{\beta} \quad (3.12)$$

$$\tilde{\nabla}_\gamma(\sqrt{-g}f'(\tilde{R})g^{\alpha\beta}) - \tilde{\nabla}_\delta(\sqrt{-g}f'(\tilde{R})g^{\delta(\alpha})\delta^{\beta)}_\gamma \quad (3.13)$$

The second field equation is obtained by varing the action with respect to the independent connection. If you remember that, under some conditions, you must recover GR, then eq. (3.13) becomes the definition of the independent Levi-Civita connection.

Note that this brings the advantage that the differential equation contains derivates none higher than second order (not as eq. (3.3) that contains fourth order derivates).

C. Metric-affine f(R) gravity

Metric-affine f(R) gravity is similar to Palatini f(R) gravity, in the sense that also assumes both, the metric tensor and the Levi-Civita connection as independent variables. The difference lies in the fact that now, we allow the matter part of the action to depend on the Levi-Civita connection as well.

This modified gravity model is still quite new and the mathematics associated to it require the introduction of several concepts and definitions. Such complications makes unviable to make an adequate treatment of the theory in this article, so we will only refer the reader to [4] in the case he or she is interested to learn more about this, but we will not go further in this particular f(R) gravity.

D. Successes and challenges

We have already introduced what f(R) gravities are, but we have not mentioned at all what are they useful for.

The main objective that f(R) gravities persecute is giving a viable alternative to dark energy theories. Many people feel uncomfortable with the idea of introducing a new fluid (in order to explain the accelerating universe) that cannot be detected or have ever been observed in any way. An alternative to avoid this, is modifying the theory of gravitation. The situation is quite similar to what happened years ago with Mercury's perihelion: Several people thought that it should be an invisible mass near Mercury when, in fact, what was happening there was that the relativistic corrections to Newtonian gravity were becoming important. Similarly, we may think that GR is not completely exact and needs some corrections on big scales, and those corrections may be given by f(R) theories of gravitation. However we must be careful about this idea. Dark energy models satisfy lots of experimental observations, and we shouldn't discard a theory simply because we feel uncomfortable about it.

There are many things that a good f(R) gravity must satisfy. For example, it must predict the early universe inflation or some other method to explain some of the observations (i.e. it must produce the same cosmological dynamics of dark energy models). It also must have a well posed Cauchy problem, meaning it must have predictive power (this may seem obvious, but in some pathological cases, the theory can't be extended in the temporal direction, given an initial condition)

Nowadays, f(R) gravities have accomplished a lot, but we are probably very far away yet to be able to claim any f(R) gravity as the correct theory. However, even if f(R) gravities turn out not to be the answer we are looking for, we shouldn't feel disappointed, since it has helped us to a better understanding of GR.

IV. CONCLUSIONS

After reading this article, hopefully, you have a better understanding of the concepts of action, lagrangian and f(R) gravity and how they work. For what is worth, nowadays, f(R) gravity has gave us some alternative to dark energy model that, in some cases, have a good fitting with experimental data, however, we are far from saying it is the right theory.

[1] E.Goldstein, Classical mechanics, third edition

[2] R.M. Wald 1984, General Relativity (Chicago University Press, Chicago)

[3] Carroll, Sean M. (2004), Spacetime and Geometry

[4] T.P.Sotiriou and V. Faraoni, arXiv:0805.1726

[5] V. Faraoni, arXiv:0810.2602, Oct. 2008