$f(R)$ gravity: successes and challenges

Valerio Faraoni

Physics Department, Bishop's University
2600 College St., Sherbrooke, Québec, Canada J1M 1Z7

Abstract

We review the state of the art of $f(R)$ theories of gravity (in their various formulations), which have been proposed as an explanation of the cosmic acceleration alternative to dark energy. The successes of $f(R)$ gravity are discussed, together with the challenges imposed by minimal criteria for their viability.

1 Introduction

The acceleration of the universe was discovered ten years ago using type Ia supernovae [1] and no definitive or truly satisfactory explanation of this phenomenon has been given yet. This discovery has important implications not only for cosmology, but also for fundamental physics. According to *WMAP* and the other experiments mapping anisotropies of the cosmic microwave background, if general relativity is the correct description of our universe, then approximately 76% of its energy content is not dark or luminous matter, but is instead a mysterious form of *dark energy*, exotic, invisible, and unclustered. Three main classes of models for this cosmic acceleration have been proposed:

1. a cosmological constant Λ

2. dark energy

3. modified gravity.

Naively, a cosmological constant propelling the cosmic acceleration and eventually coming to dominate the universe, causing it to enter a de Sitter phase without return, seems the most obvious explanation. However, Λ brings with it the notorious cosmological constant problem and the coincidence problem. To admit that Λ is non-zero but still has the tiny value required to cause the current acceleration amounts to an extreme fine-tuning. It is not surprising, therefore, that most cosmologists reject this explanation, postulating that Λ is exactly zero for reasons yet to be discovered (the cosmological constant problem is put in a drawer for the time being), and that a different explanation is to be found for the cosmic acceleration.

The second class of models, (mostly) within the context of general relativity, postulates the existence of a dark energy fluid with equation of state $P \approx -\rho$ (where ρ and P are the energy density and pressure of the fluid, respectively), which comes to dominate late in the matter era. Dark energy could even be phantom energy with equation of state such that $P < -\rho$. Many dark energy models have been studied, none of which is totally convincing or free of fine-tuning problems, or can be demonstrated to be the "correct" one.

A third possibility consists in dispensing entirely with the mysterious dark energy and modifying gravity at the largest scales.[1] Here we focus on modified gravity and, specifically, on the so-called $f(R)$ gravity theories.[2]

$f(R)$ or "modified" gravity consists of infrared modifications of general relativity that become important only at low curvatures, late in the matter era. The Einstein-Hilbert action[3] $S_{EH} = \frac{1}{2\kappa} \int d^4x \sqrt{-g}\, R + S^{(matter)}$ is modified to

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g}\, f(R) + S^{(matter)} \, , \tag{1.1}$$

where $f(R)$ is a non-linear function of its argument [3, 4].

In principle, the metric tensor contains several degrees of freedom: tensor, vector, and scalar, massless or massive. In general relativity only the familiar massless spin 2 graviton propagates. When the Einstein-Hilbert action is modified, other degrees of freedom appear. The change

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[1]Historically, this approach was the correct one in explaining the precession of Mercury's perihelion: not due to an unseen mass but to Einstein's modification of Newtonian gravity.

[2]We do not consider here braneworld models or other modifications of gravity that are sometimes also referred to as "modified gravity".

[3]Here R is the Ricci curvature of the metric tensor g_{ab} with metric determinant g, $\kappa \equiv 8\pi G$, G is Newton's constant, and we follow the notations of Ref. [2].

$R - 2\Lambda \to f(R)$ in the action brings to life, in addition to the massless graviton, a massive scalar mode which can drive the cosmic acceleration, and will be discussed below. This is analogous to the inflaton field driving the accelerated expansion of the early universe, although at a much lower energy scale.

If terms quadratic in the Ricci and Riemann tensor, and possibly other curvature invariants, are included in the gravitational Lagrangian, $f\left(R, R_{ab}R^{ab}, R_{abcd}R^{abcd}, ...\right)$, massive gravitons and vector degrees of freedom appear. In the following we restrict ourselves to $f(R)$ theories of gravity in their various formulations, and we focus on their use as substitutes for dark energy;[4] for a more comprehensive discussion we refer the reader to [7] and for short introductions to [8, 9].

We adopt a conservative point of view and regard $f(R)$ gravity more as a toy model than the correct theory of gravity, *i.e.*, we consider these theories as a proof of principle that modifying gravity is a viable alternative to dark energy. However, we do not feel that one can claim that any of the $f(R)$ models proposed thus far is the "correct" one, or has exceptional support from the observational data. While it is true that many $f(R)$ models pass all the available experimental tests and fit the cosmological data, the same is true for many dark energy models, and it is currently impossible to use observational data to discriminate between most of them, and between dark energy and modified gravity models.

Modifying gravity is risky: unwanted consequences may be violations of the experimental limits on the parametrized-post-Newtonian (PPN) parameters at terrestrial and Solar System scales [10], instabilities, ghosts and, as in any newly proposed theory, the Cauchy problem could be ill-posed. These aspects are discussed in the next sections.

$f(R)$ gravity has a long history: its origins can be loosely traced to Weyl's 1919 theory in which a term quadratic in the Weyl tensor was added to the Einstein-Hilbert Lagrangian [11]. Later, $f(R)$ gravity received the attention of many authors, including Eddington, Bach, Lanczos, Schrödinger, and Buchdahl. In the 1960's and 1970's, it was found that quadratic corrections to S_{EH} were necessary to improve the renormalizability of general relativity [12], and in 1980 quadratic corrections were found to fuel inflation without the need for scalar fields [13]. Non-linear corrections are also motivated by string theories [14]. We refer the reader to [15] for an historical review.

The prototype of $f(R)$ gravity [3, 4] is the model

$$f(R) = R - \mu^4/R \,, \tag{1.2}$$

where μ is a mass scale of the order of the present value of the Hubble parameter $\mu \sim H_0 \sim 10^{-33}$ eV. Although ruled out by its weak-field limit [16] and by a violent instability [17], this model gives the idea underlying modified gravity: the $1/R$ correction is negligible in comparison with R at the high curvatures of the early universe, and kicks in only as $R \to 0$, late in the history of the universe.

Many forms of the function $f(R)$ are found in the literature: here we discuss only general, model-independent, features of $f(R)$ gravity.

2   The three versions of $f(R)$ gravity

Modified gravity comes in three versions:
1) metric (or second order) formalism;

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[4] $f(R)$ gravity, sometimes with an explicit coupling of matter to R [5], has also been used as an alternative to galactic dark matter [6].

2) Palatini (or first order) formalism; and

3) metric-affine gravity.

2.1 Metric $f(R)$ gravity

In the metric formalism [3, 4], the action is

$$S_{metric} = \frac{1}{2\kappa} \int d^4x \, \sqrt{-g} \, f(R) + S^{(matter)} \, . \tag{2.1}$$

Variation with respect to the (inverse) metric tensor g^{ab} yields the field equation

$$f'(R)R_{ab} - \frac{f(R)}{2} g_{ab} = \nabla_a \nabla_b f'(R) - g_{ab} \Box f'(R) + \kappa T_{ab} \, , \tag{2.2}$$

where a prime denotes differentiation with respect to R. The first two terms on the right hand side introduce fourth order derivatives of the metric, hence the name "fourth order gravity" sometimes given to these theories.

The trace of eq. (2.2) yields

$$3\Box f'(R) + Rf'(R) - 2f(R) = \kappa T \, , \tag{2.3}$$

where $T \equiv T^a{}_a$ is the trace of the matter stress-energy tensor. This second order differential equation for $f'(R)$ differs deeply from the trace of the Einstein equation $R = -\kappa T$ which, instead, relates algebraically the Ricci scalar to T. We already see that $f'(R)$ is indeed a dynamical variable, the scalar degree of freedom contained in the theory.

Formally, one can rewrite the field equation (2.2) in the form of an effective Einstein equation as

$$G_{ab} = \kappa \left(T_{ab} + T_{ab}^{(eff)} \right) \tag{2.4}$$

where

$$T_{ab}^{(eff)} = \frac{1}{\kappa} \left[\frac{f(R) - Rf'(R)}{2} g_{ab} + \nabla_a \nabla_b f'(R) - g_{ab} \Box f'(R) \right] \tag{2.5}$$

is an effective stress-energy tensor containing geometric terms. Of course, as usual when adopting this procedure, $T_{ab}^{(eff)}$ does not satisfy any energy condition and the effective energy density is, in general, not positive-definite. As is clear from these equations, in $f(R)$ gravity one can define an effective gravitational coupling $G_{eff} \equiv G/f'(R)$ in a way analogous to what is done in scalar-tensor theories. Hence, $f'(R)$ must be positive in order for the graviton to carry positive kinetic energy.

In the spatially flat Friedmann-Lemaitre-Robertson-Walker (FLRW) metric adopted as the kinematic description of our universe,

$$ds^2 = -dt^2 + a^2(t) \left(dx^2 + dy^2 + dz^2 \right) \, , \tag{2.6}$$

the field equations of metric $f(R)$ cosmology assume the form

$$H^2 = \frac{\kappa}{3f'(R)} \left[\rho^{(matter)} + \frac{Rf'(R) - f(R)}{2} - 3H\dot{R}f''(R) \right] \, , \tag{2.7}$$

$$2\dot{H} + 3H^2 = -\frac{\kappa}{f'(R)} \left[P^{(matter)} + f'''(R) \left(\dot{R} \right)^2 + 2H\dot{R}f''(R) + \ddot{R}f''(R) \right.$$

$$\left. + \frac{f(R) - Rf'(R)}{2} \right] \, , \tag{2.8}$$

where an overdot denotes differentiation with respect to the comoving time t. The corresponding phase space is a 2-dimensional curved manifold embedded in a 3-dimensional space and with a rather complicated structure [18].

2.2 Palatini $f(R)$ gravity

In the Palatini approach, both the metric g_{ab} and the connection Γ^a_{bc} are independent variables, *i.e.*, the connection is not the metric connection of g_{ab}. While in general relativity the metric and Palatini variations produce the same (Einstein) equations, this is no longer true for non-linear Lagrangians.[5]

Shortly after metric $f(R)$ theories were proposed as alternatives to dark energy, also the Palatini version was adanced for the same purpose, originally in its $f(R) = R - \mu^4/R$ incarnation [20]. The Palatini action is

$$S_{Palatini} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \, f\left(\tilde{R}\right) + S^{(matter)} \left[g_{ab}, \psi^{(m)}\right] \; . \tag{2.9}$$

There are two Ricci tensors: R_{ab}, which is constructed using the metric connection of the (unique) physical metric g_{ab}, and $\tilde{R}_{ab}$ which is the Ricci tensor of the non-metric connection Γ^a_{bc}. The latter gives rise to the scalar $\tilde{R} \equiv g^{ab}\tilde{R}_{ab}$. The matter part of the action does not depend explicitly from the connection Γ, but only from the metric and the matter fields, collectively denoted with $\psi^{(m)}$.

Variation of the Palatini action (2.9) yields the field equation

$$f'(\tilde{R})\tilde{R}_{ab} - \frac{f(\tilde{R})}{2} \, g_{ab} = \kappa \, T_{ab} \; . \tag{2.10}$$

Note the absence of second covariant derivatives of f', in contrast with eq. (2.2). Variation with respect to the independent connection produces the field equation

$$\tilde{\nabla}_d \left(\sqrt{-g} \, f'(\tilde{R}) g^{ab}\right) - \tilde{\nabla}_d \left(\sqrt{-g} \, f'(\tilde{R}) g^{d(a}\right) \delta^{b)}_c = 0 \; , \tag{2.11}$$

where $\tilde{\nabla}_c$ denotes the covariant derivative associated to this non-metric connection Γ. The trace of eqs. (2.10) and (2.11) yields

$$f'(\tilde{R})\tilde{R} - 2f(\tilde{R}) = \kappa \, T \tag{2.12}$$

and

$$\tilde{\nabla}_c \left(\sqrt{-g} \, f'(\tilde{R}) g^{ab}\right) = 0 \; . \tag{2.13}$$

Eq. (2.13) tells us that $\tilde{\nabla}_c$ is the covariant derivative of the metric

$$\tilde{g}_{ab} \equiv f'(\tilde{R}) g_{ab} \tag{2.14}$$

conformally related to g_{ab}. Note that eq. (2.12) is an algebraic (or trascendental, depending on the form of the function f) and not a differential equation for $f'(\tilde{R})$: hence, this quantity is non-dynamical, contrary to metric $f(R)$ gravity. This lack of dynamics has important consequences explored in the following sections. It is possible to eliminate completely the non-metric

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[5]The requirement that the Palatini and metric variations give the same field equations selects Lovelock gravity [19], of which general relativity is a special case.

connection from the field equations, which are then rewritten as

$$G_{ab} = \frac{\kappa}{f'} T_{ab} - \frac{1}{2} \left(R - \frac{f}{f'} \right) g_{ab} + \frac{1}{f'} \left(\nabla_a \nabla_b - g_{ab} \Box \right) f'$$

$$- \frac{3}{2(f')^2} \left[\nabla_a f' \nabla_b f' - \frac{1}{2} g_{ab} \nabla_c f' \nabla^c f' \right] . \tag{2.15}$$

2.3 Metric-affine $f(R)$ gravity

In metric-affine $f(R)$ gravity [21], also the matter part of the action

$$S_{affine} = \frac{1}{2\kappa} \int d^4 x \sqrt{-g} \, f \left(\tilde{R} \right) + S^{(matter)} \left[g_{ab}, \Gamma^a_{bc}, \psi^{(m)} \right] , \tag{2.16}$$

depends explicitly on the connection Γ, which is possibly non-symmetric. This leads to a torsion associated with matter, and to a modern revival of torsion theories. These were originally introduced within a non-cosmological context, with the spin of elementary particles coupling to the torsion. Metric-affine $f(R)$ gravity has not yet been explored in great detail, especially with respect to its cosmological consequences. For these reasons, in the following we focus on metric and Palatini $f(R)$ gravity.

3 Equivalence of metric and Palatini $f(R)$ gravities with Brans-Dicke theories

If $f''(R) \neq 0$, metric modified gravity is equivalent to an $\omega = 0$ Brans-Dicke theory[6] [22], while Palatini modified gravity is equivalent to an $\omega = -3/2$ one. This equivalence has been proposed and rediscovered, for particular theories or in general, many times over the years [23].

3.1 Metric formalism

Assuming that $f''(R) \neq 0$ and beginning with the action (1.1), one introduces the auxiliary scalar field $\phi = R$ and considers the action

$$S = \frac{1}{2\kappa} \int d^4 x \sqrt{-g} \left[\psi(\phi) R - V(\phi) \right] + S^{(matter)} , \tag{3.1}$$

where

$$\psi(\phi) = f'(\phi) , \qquad V(\phi) = \phi f'(\phi) - f(\phi) . \tag{3.2}$$

The action (3.1) trivially reduces to (1.1) for metric $f(R)$ gravity if $\phi = R$. Vice-versa, the variation of (3.1) with respect to g^{ab} gives

$$G_{ab} = \frac{1}{\psi} \left(\nabla_a \nabla_b \psi - g_{ab} \Box \psi - \frac{V}{2} g_{ab} \right) + \frac{\kappa}{\psi} T_{ab} , \tag{3.3}$$

while varying with respect to ϕ yields

$$R \frac{d\psi}{d\phi} - \frac{dV}{d\phi} = (R - \phi) f''(\phi) = 0 \tag{3.4}$$

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[6]The general form of the Brans-Dicke action is $S_{BD} = \frac{1}{2\kappa} \int d^4 x \sqrt{-g} \left[\phi R - \frac{\omega}{\phi} \nabla^c \phi \nabla_c \phi - V(\phi) \right] + S^{(matter)}$.

and $\phi = R$ under the assumption $f'' \neq 0$. Hence, the scalar field $\phi = R$ is dynamical and satisfies the trace equation

$$3f''(\phi)\Box\phi + 3f'''(\phi)\nabla^c\phi\nabla_c\phi + \phi f'(\phi) - 2f(\phi) = \kappa T \, . \tag{3.5}$$

This scalar is massive: as discussed later, the analysis of small perturbations of de Sitter space allows one to compute explicitly its mass squared

$$m_\phi^2 = \frac{1}{3}\left(\frac{f_0'}{f_0''} - R_0\right) \, , \tag{3.6}$$

where a zero subscript denotes quantities evaluated at the constant curvature of the de Sitter background. It turns out to be more convenient to consider the scalar $\psi \equiv f'(\phi)$, which satisfies

$$3\Box\psi + 2U(\psi) - \psi\frac{dU}{d\psi} = \kappa T \tag{3.7}$$

with $U(\psi) = V(\phi(\psi)) - f(\phi(\psi))$. It is clear, therefore, that the theory contains a scalar degree of freedom, and the action

$$S = \frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\,[\psi R - U(\psi)] + S^{(matter)} \, , \tag{3.8}$$

is recognized as an $\omega = 0$ Brans-Dicke theory. This theory, called "massive dilaton gravity" was originally introduced in the 1970's in order to generate a Yukawa term in the Newtonian limit [24]. The assumption $f'' \neq 0$ can be seen as the requirement that the change of variable $R \to \psi(R)$ be invertible.

3.2 Palatini formalism

In the Palatini case, the discussion of the equivalence with a Brans-Dicke theory proceeds in a way analogous to that of the metric formalism. One begins with the action (2.9) and introduces $\phi = \tilde{R}$ and $\psi \equiv f'(\phi)$. Then, apart from a boundary term that can be neglected for classical purposes, the action is rewritten, in terms of the metric g_{ab} and of its Ricci tensor R_{ab}, as

$$S_{Palatini} = \frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\,\left[\psi R + \frac{3}{2\psi}\nabla^c\psi\nabla_c\psi - V(\psi)\right] + S^{(matter)} \, , \tag{3.9}$$

where we used the fact that, since $\tilde{g}_{ab} = \psi\,g_{ab}$, the Ricci curvatures of g_{ab} and $\tilde{g}_{ab}$ are related by

$$\tilde{R} = R + \frac{3}{2\psi}\nabla^c\psi\nabla_c\psi - \frac{3}{2}\Box\psi \, . \tag{3.10}$$

The action (3.9) is recognized as a Brans-Dicke theory with Brans-Dicke parameter $\omega = -3/2$.

4 Criteria for viability

In order for $f(R)$ gravity to be successful, it is not sufficient that it serves the purpose for which it was introduced in the cosmological context, but it must also pass the tests imposed by Solar System and terrestrial experiments on relativistic gravity, and it must satisfy certain minimal criteria for viability. Overall, these are:

- possess the correct cosmological dynamics;

- not suffer from instabilities and ghosts;

- have the correct Newtonian and post-Newtonian limit;

- give rise to cosmological perturbations compatible with the data from the cosmic microwave background and large scale structure surveys; and

- have a well-posed Cauchy problem.

The failure to satisfy even a single one of these criteria is taken as a statement that the theory is doomed. These viability criteria are examined in the following.

4.1 Correct cosmological dynamics

In the opinion of most cosmologists, in order to be acceptable a cosmological model must exhibit early inflation (or an alternative way to solve the horizon, flatness, and monopole problem together with a mechanism to generate density perturbations), followed by a radiation-dominated era and a matter-dominated era, and then by the present accelerated epoch that $f(R)$ theories were resurrected to explain. The future era is usually found to be an eternal de Sitter attractor phase, or a Big Rip singularity truncating the history of the universe at a finite time.

Smooth transitions between different eras are required. It has been pointed out that the exit from the radiation era, in particular, may have problems in many models [25], a warning that care must be exerted in building $f(R)$ cosmologies. Ultimately, exit from the radiation, or any era can be achieved. Take, for example, what we could name "designer $f(R)$ gravity": one can prescribe a desired expansion history of the universe by a choice of the scale factor $a(t)$ and then integrate an ODE that determines the function $f(R)$ that produces $a(t)$ [26]. In general, this function is not unique and assumes rather contrived forms (not the usual $R - \mu^{2(n+1)}/R^n$, or simple forms like that).

4.2 Instabilities

The prototype model in the discussion of instabilities is again the choice $f(R) = R - \mu^4/R$ with $\mu \sim H_0 \sim 10^{-33}$ eV. Shortly after it was proposed, this model was found to suffer from a catastrophic ("Dolgov-Kawasaki") instability [17]. The stability analysis was later generalized to any metric $f(R)$ theory [27] and the extension to even more general gravitational theories has been pursued [28]. One proceeds by parametrizing the deviations from general relativity as

$$f(R) = R + \epsilon\varphi(R) \,, \tag{4.1}$$

where ϵ is a small positive constant with the dimensions of a mass squared and the function φ is dimensionless. The trace equation for the Ricci scalar R takes the form

$$\Box R + \frac{\varphi'''}{\varphi''} \nabla^c R \nabla_c R + \left(\frac{\epsilon\varphi' - 1}{3\epsilon\varphi''}\right) R = \frac{\kappa T}{3\epsilon\varphi''} + \frac{2\varphi}{3\varphi''} \,. \tag{4.2}$$

Next, one expands around a de Sitter background and writes the metric *locally* as

$$g_{ab} = \eta_{ab} + h_{ab} \,, \tag{4.3}$$

while the scalar degree of freedom R is expanded as

$$R = -\kappa T + R_1 \,, \tag{4.4}$$

with R_1 a perturbation. To first order, the trace equation yields the dynamical equation for R_1

$$\ddot{R}_1 - \nabla^2 R_1 - \frac{2\kappa\varphi'''}{\varphi''} \dot{T}\dot{R}_1 + \frac{2\kappa\varphi'''}{\varphi''} \vec{\nabla}T \cdot \vec{\nabla}R_1 + \frac{1}{3\varphi''}\left(\frac{1}{\epsilon} - \varphi'\right)R_1 = \kappa\ddot{T} - \kappa\nabla^2 T - \frac{\left(\kappa T\varphi^2 + 2\varphi\right)}{3\varphi''} \ .$$
(4.5)

The last term on the left hand side is dominated by the term in ϵ^{-1} and gives the effective mass squared of R_1

$$m^2 \simeq \frac{1}{3\epsilon\varphi''} \ ,$$
(4.6)

from which one deduces that the theory is

- *stable* if $f''(R) > 0$

- *unstable* if $f''(R) < 0$.

The case of general relativity is excluded by the assumption $f'' \neq 0$, but the well-known stability in this case allows one to extend the stability criterion for metric $f(R)$ gravity to be $f'' \geq 0$.

As an example, the prototype model $f(R) = R - \mu^4/R$, which has $f'' < 0$ is unstable. The time scale for the onset of this instability is dictated by the smallness of the scale μ and is seen to correspond to $\sim 10^{-26}$ s [17], making this an explosive instability.

One can give a physical interpretation of this result as follows [29]: remembering that the effective gravitational coupling is $G_{eff} = G/f'(R)$, if $dG_{eff}/dR = -f''G/(f')^2 > 0$ (which corresponds to $f'' < 0$), then G_{eff} increases with R and a large curvature causes gravity to become stronger, which in turn causes a larger R, in a positive feedback mechanism driving the system away. If instead $dG_{eff}/dR < 0$, then a negative feedback damps the increase in the gravitational coupling strength.

Palatini $f(R)$ gravity, by contrast, is described by second order field equations, the trace equation $f'(\tilde{R})\tilde{R} - 2f(\tilde{R}) = \kappa T$ is not a differential equation but rather a non-dynamical algebraic one and, therefore, there is no Dolgov-Kawasaki instability [30].

The previous analysis for metric $f(R)$ gravity obtained with the local expansion (4.3) is necessarily limited to short wavelengths (compared to the curvature radius), but can be extended to the longest wavelenghts [31]. This is necessarily more complicated because these modes suffer from the notorious gauge-dependence problems of cosmological perturbations and a covariant and gauge-invariant formalism is needed. We assume that the background space is de Sitter and consider the general action

$$S = \int d^4x \sqrt{-g} \left[\frac{f(\phi, R)}{2} - \frac{\omega(\phi)}{2} \nabla^c\phi\nabla_c\phi - V(\phi)\right] \ ,$$
(4.7)

which contains $f(R)$ and scalar-tensor gravity, and mixtures of them. On a FLRW background, the field equations become

$$H^2 = \frac{1}{3f'}\left(\frac{\omega}{2}\dot{\phi}^2 + \frac{Rf'-f}{2} + V - 3H\dot{f}\right) \ ,$$
(4.8)

$$\dot{H} = \frac{-1}{2f'}\left(\omega\dot{\phi}^2 + \ddot{f}' - H\dot{f}'\right) \ ,$$
(4.9)

$$\ddot{\phi} + 3H\dot{\phi} + \frac{1}{2\omega}\left(\frac{d\omega}{d\phi}\dot{\phi}^2 - \frac{\partial f}{\partial\phi} + 2\frac{dV}{d\phi}\right) = 0 \ .$$
(4.10)

de Sitter space is a solution subject to the conditions

$$6H_0^2 f_0' - f_0 + 2V_0 = 0 \,, \qquad f_0' = 2V_0' \,. \tag{4.11}$$

An analysis [31] using the covariant and gauge-invariant Bardeen-Ellis-Bruni-Hwang formalism [32] in the version given by Hwang [33] for alternative gravitational theories yields the stability condition of de Sitter space in metric $f(R)$ gravity with respect to *inhomogeneous* perturbations

$$\frac{(f_0')^2 - 2f_0 f_0''}{f_0' f_0''} \geq 0 \,, \tag{4.12}$$

which is obtained in the zero momentum limit. This condition coincides with the stability condition with respect to *homogeneous* perturbations [29].

At this point it is worth checking that the equivalence between metric $f(R)$ gravity and an $\omega = 0$ Brans-Dicke theory holds also at the level of perturbations; previous doubts to this regard [34, 35] have now been dissipated.

For the $\omega = 0$ Brans-Dicke theory, the stability condition of de Sitter space with respect to inhomogeneous perturbations is given again by eq. (4.12), while that for stability with respect to homogeneous perturbations is

$$\frac{(f_0')^2 - 2f_0 f_0''}{f_0'} \geq 0 \,. \tag{4.13}$$

This is again equivalent to (4.12) provided that stability against *local* perturbations, expressed by $f_0'' > 0$, is assumed. Therefore, there is complete equivalence between metric $f(R)$ gravity and $\omega = 0$ Brans-Dicke theory also at the level of cosmological perturbations.

Going beyond the linear approximation, metric $f(R)$ theories have been found to be susceptible to another, non-linear, instability, which makes it hard to build models of relativistic stars in strong gravity. In these situations, a singularity develops for large R, which was discovered in [36]. Although this problem needs further study, it seems that, in order to avoid this singularity requires some degree of fine-tuning. At present, this is probably the biggest challenge for metric $f(R)$ theories.

4.3 Ghosts

Ghosts are massive states of negative norm which cause lack of unitarity and are common when trying to generalize Einstein's gravity. The good news here are that $f(R)$ gravity is ghost-free. More general theories of the form $f\left(R, R_{ab}R^{ab}, R_{abcd}R^{abcd}, ...\right)$, in general, contain ghost fields. A possible exception (under certain conditions [37]) is the case in which the extra terms appear in the Gauss-Bonnet combination $\mathcal{G} = R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}$, as in $f = f(R, \mathcal{G})$ In this case, the field equations are of second order and there are no ghosts [38, 39, 40].

4.4 Weak-field limit (metric formalism)

Early work on the weak-field limit of both metric and Palatini $f(R)$ gravity was subject to errors and incompleteness (see [7] for details[7]); a satisfactory treatment for the prototype model $f(R) = R - \mu^4/R$ in the metric formalism was given in [42] and then generalized to arbitrary forms of the function $f(R)$ in [43, 16].

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[7]The limit of $f(R)$ gravity to general relativity has the character of a singular limit, and even the limit to general relativity of Brans-Dicke theory is not free from ambiguities [41].

In order to assess whether the limits set on the PPN parameter γ by the available Solar System experiments, one needs to find the weak-field solution of the field equations and compute this parameter. One considers a static, spherically symmetric, non-compact body which constitutes a perturbation of a background de Sitter universe. The line element is written as

$$ds^2 = -\left[1 + 2\Psi(r) - H_0^2 r^2\right] dt^2 + \left[1 + 2\Phi(r) + H_0^2 r^2\right] dr^2 + r^2 d\Omega^2 \qquad (4.14)$$

in Schwarzschild coordinates, where $d\Omega^2$ is the line element on the unit 2-sphere and Ψ and Φ are post-Newtonian potentials. These are of small amplitude, $|\Psi(r)|, |\Phi(r)| << 1$, and one considers small (non-cosmological) scales so that $H_0 r << 1$, while expanding the Ricci scalar around the constant curvature of the background de Sitter space, $R(r) = R_0 + R_1$. The PPN parameter γ is given by $\gamma = -\Phi(r)/\Psi(r)$ [10]. Three assumptions are made [16]:

1. $f(R)$ is analytical at R_0;

2. $mr << 1$, where m is the effective mass of the scalar degree of freedom of the theory, *i.e.*, it is assumed that this scalar field is light and has a range larger than the size of the Solar System (we remind the reader that there are no experimental constraints of scalars with range $m^{-1} < 0.2$ mm).

3. For the matter composing the spherical body, the pressure is negligible, $P \simeq 0$, so that $T = T_0 + T_1 \simeq -\rho$.

The first and the last assumption are not stringent, but the second one is, as will be clear below. The trace equation (2.3) yields the equation for the Ricci scalar perturbation

$$\nabla^2 R_1 - m^2 R_1 = \frac{-\kappa\rho}{3f_0''} , \qquad (4.15)$$

where

$$m^2 = \frac{(f_0')^2 - 2f_0 f_0''}{3f_0' f_0''} \qquad (4.16)$$

is the effective mass squared of the scalar. Eq. (4.16) coincides with the expression obtained in the gauge-invariant stability analysis of de Sitter space and in propagator calculations.

If $mr << 1$, the solution of the linearized field equations is

$$\Psi(r) = \frac{-\kappa M}{6\pi f_0'} \frac{1}{r} , \qquad (4.17)$$

$$\Phi(r) = \frac{\kappa M}{12\pi f_0'} \frac{1}{r} . \qquad (4.18)$$

The PPN parameter sought for is, therefore,

$$\gamma = \frac{-\Phi(r)}{\Psi(r)} = \frac{1}{2} , \qquad (4.19)$$

in gross violation of the (recently improved) experimental limit [44]

$$|\gamma - 1| < 2.3 \cdot 10^{-5} . \qquad (4.20)$$

This result would be the end of metric $f(R)$ gravity if the assumptions made in the calculation were satisfied. However, this is not the case for assumption 2): mr is not always less than unity

due to the *chameleon effect*. This consists in the effective mass m depending on the curvature or, alternatively, the matter density of the environment. The scalar degree of freedom can be short-ranged (say $m > 10^{-3}$ eV, corresponding to a range $\lambda < 0.2$ mm) at Solar System densities and evade the experimental constraints, while being long-ranged at cosmological densities and thus being able to affect the cosmological dynamics [40, 45]. Although at a first glance the chameleon effect could be seen as a contrived and fine-tuned mechanism, $f(R)$ gravity is rather complicated and the effective range does indeed depend on the environment. The chameleon mechanism is well-known and accepted in quintessence models, in which it was discovered for the scalar field potential $V(\phi) \approx 1/\phi$ [46]. Many forms of the function $f(R)$ are known to exhibit the chameleon mechanism and pass the observational tests. For example, the model

$$f(R) = R - (1 - n) \, \mu^2 \left(\frac{R}{\mu^2} \right)^n \tag{4.21}$$

is compatible with the PPN limits if $\mu \sim 10^{-50}$ eV$\sim 10^{-17} H_0$ [45]. It is obvious that a correction term $\sim R^n$ with $n < 1$ to the Einstein-Hilbert Lagrangian R will come to dominate as $R \to 0^+$ (for example, for $n = 1/2$, $\sqrt{R} > R$ as $R \to 0$). The model (4.21) is compatible with the experimental data but it could be essentially indistinguishable from a dark energy model. Hope of discriminating between dark energy and $f(R)$ models, or between different modified gravities relies on the study of the growth of cosmological perturbations.

4.5   Correct dynamics of cosmological perturbations

The expansion history of the universe alone is not sufficient to discriminate between various models, but the growth of structures depends on the theory of gravity and has the potential to achieve this goal. Song, Hu, and Sawicki [47] assumed an expansion history $a(t)$ typical of the ΛCDM model and found that vector and tensor modes are not affected by $f(R)$ corrections to Einstein gravity, to lowest order, while scalar modes are. They also found the condition $f''(R) > 0$ for the stability of scalar perturbations, in agreement with the arguments discussed above. The most interesting results are that $f(R)$ corrections lower the large angle anisotropies of the cosmic microwave background and produce correlations between cosmic microwave background and galaxy surveys different from those of dark energy models.

Overall, the study of structure formation in modified gravity is still work in progress, and often is performed within the context of specific models, some of which are already in trouble because they do not pass the weak-field limit or the stability constraints. A similar situation holds for all Palatini $f(R)$ models, and for this reason, their weak-field limit and cosmological perturbations are not discussed here.

4.6   The Cauchy problem

A physical theory must have predictive power and, therefore, a well-posed initial value problem. General relativity satisfies this requirement for "reasonable" forms of matter [2]. The well-posedness of the Cauchy problem for vacuum $f(R)$ gravity was briefly discussed for special metric models in earlier papers [48]. Thanks to the equivalence between $f(R)$ gravity and scalar-tensor theory when $f''(R) \neq 0$, the Cauchy problem can be reduced to the analogous one for Brans-Dicke gravity with $\omega = 0, -3/2$. That the initial value problem is well-posed was demonstrated for particular scalar-tensor theories in [49, 48] and a general analysis has only recently been performed [50, 51]. This work, however, does not cover the $\omega = 0, -3/2$ cases.

A system of $3+1$ equations of motion is *well-formulated* if it can be written as a system of equations of only first order in both time and space derivatives. If the latter is cast in the full first order form

$$\partial_t \vec{u} + M^i \nabla_i \vec{u} = \vec{S}(\vec{u}), \tag{4.22}$$

where $\vec{u}$ collectively denotes the fundamental variables h_{ij}, K_{ij}, *etc.* introduced below, M^i is called the *characteristic matrix* of the system, and $\vec{S}(\vec{u})$ describes source terms and contains only the fundamental variables but not their derivatives. The initial value formulation is *well-posed* if the system of PDEs is *symmetric hyperbolic* (*i.e.*, the matrices M^i are symmetric) and *strongly hyperbolic* if $s_i M^i$ has a real set of eigenvalues and a complete set of eigenvectors for any 1-form s_i, and obeys some boundedness conditions [52].

In short, the result obtained in [53] is that the Cauchy problem for metric $f(R)$ gravity is well-formulated and is well-posed in vacuo and with "reasonable" forms of matter (*i.e.*, perfect fluids, scalar fields, or the Maxwell field) while for Palatini $f(R)$ gravity, instead, the Cauchy problem is not well-formulated nor well-posed due to the presence of higher derivatives of the matter fields in the field equations and to the fact that it is impossible to eliminate them [53].

Let us consider the scalar-tensor action

$$S = \int d^4x \sqrt{-g} \left[\frac{f(\phi)R}{2\kappa} - \frac{1}{2}\nabla^c \phi \nabla_c \phi - V(\phi) \right] + S^{(matter)} ; \tag{4.23}$$

Salgado [50] showed that the corresponding Cauchy problem is well-posed in vacuo and well-formulated otherwise. With the exception of $\omega = -3/2$, Salgado's results can be extended to the more general action

$$S = \int d^4x \sqrt{-g} \left[\frac{f(\phi)R}{2\kappa} - \frac{\omega(\phi)}{2}\nabla^c \phi \nabla_c \phi - V(\phi) \right] + S^{(matter)} , \tag{4.24}$$

which contains the additional coupling function $\omega(\phi)$ [53].

Setting $\kappa = 1$ in this section and performing a $3+1$ Arnowitt-Deser-Misner splitting, one introduces lapse N, shift N^i, spatial metric h_{ij}, extrinsic curvature K_{ij}, and spatial gradient Q_i of ϕ [2, 55, 50]. Assume that a time function t exists such that the spacetime (M, g_{ab}) admits a foliation with hypersurfaces Σ_t of constant t with unit timelike normal n^a. The 3-metric and projection operator on Σ_t are $h_{ab} = g_{ab} + n_a n_b$ and $h^a{}_b$, respectively, with

$$n^c n_c = -1, \quad h_{ab}n^b = h_{ab}n^a = 0, \quad h_a{}^b h_{bc} = h_{ac} , \tag{4.25}$$

$$\tag{4.26}$$

$$ds^2 = -\left(N^2 - N^i N_i\right)dt^2 - 2N_i dt dx^i + h_{ij}dx^i dx^j \tag{4.27}$$

$(i, j = 1, 2, 3)$, while $Q_c \equiv D_c\phi$, and the momentum of ϕ is $\Pi = \mathcal{L}_n\phi = n^c \nabla_c \phi$. Moreover,

$$K_{ij} = -\nabla_i n_j = -\frac{1}{2N}\left(\frac{\partial h_{ij}}{\partial t} + D_i N_j + D_j N_i\right), \tag{4.28}$$

$$\Pi = \frac{1}{N}\left(\partial_t \phi + N^c Q_c\right), \tag{4.29}$$

$$\partial_t Q_i + N^l \partial_l Q_i + Q_l \partial_i N^l = D_i(N\Pi). \tag{4.30}$$

Omitting the calculations, the reduced $3 + 1$ equations are [53]

$$\partial_t K^i{}_j + N^l \partial_l K^i{}_j + K^i{}_l \partial_j N^l - K_j{}^l \partial_l N^i + D^i D_j N$$
$$-^{(3)} R^i{}_j N - N K K^i{}_j + \frac{N}{2\phi} \delta^i_j \left[2V(\phi) + \Box\phi \right]$$
$$+ \frac{N}{\phi} \left(D^i Q_j + \Pi K^i{}_j \right) + \frac{N\omega_0}{\phi^2} Q^i Q_j$$
$$= \frac{N}{2\phi} \left[\left(S^{(m)} - E^{(m)} \right) \delta^i_j - 2 S^{(m)\,i}{}_j \right] , \tag{4.31}$$

$$\partial_t K + N^l \partial_l K +^{(3)} \Delta N - N K_{ij} K^{ij}$$
$$- \frac{N}{\phi} \left(D^\nu Q_\nu + \Pi K \right) - \frac{\omega_0 N}{\phi^2} \Pi^2$$
$$= \frac{N}{2\phi} \left[-2V(\phi) - 3\Box\phi + S^{(m)} + E^{(m)} \right] , \tag{4.32}$$

$$\left(\omega_0 + \frac{3}{2} \right) \Box\phi = \frac{T^{(m)}}{2} - 2V(\phi) + \phi V'(\phi) + \frac{\omega_0}{\phi} \left(\Pi^2 - Q^2 \right) . \tag{4.33}$$

Note that second derivatives of the scalar ϕ appear only in the form $\Box\phi$. For $\omega_0 = 0$ Brans-Dicke theory, equivalent to metric $f(R)$ gravity, one can use eq. (4.33) to eliminate completely the d'Alembertian $\Box\phi$ from the remaining equations. As a result, the Cauchy problem is well-formulated in general and well-posed in vacuo. Further work by Salgado and collaborators [51] established the well-posedness of the Cauchy problem for scalar-tensor gravity with $\omega = 1$ in the presence of matter, which implies well-posedness for metric $f(R)$ gravity with matter along the lines established above.

Palatini $f(R)$ gravity, instead, is equivalent to an $\omega_0 = -3/2$ Brans-Dicke theory: for this value of the Brans-Dicke parameter, the d'Alembertian $\Box\phi$ disappears from eq. (4.33) and the field ϕ is not dynamical (there is no wave equation to govern it): it can be specified arbitrarily on a spacetime region provided that its gradient obeys the constraint (4.33). Hence, in Palatini $f(R)$ gravity it is impossible to eliminate $\Box\phi$ from the system of differential equations unless, of course, $\Box\phi = 0$ (including the case of general relativity if $\phi = $ constant). Apart from the impossibility of a first-order formulation, one sees that for $\omega = -3/2$ the dynamical wave equation for ϕ is lost completely and this field is non-dynamical. Palatini $f(R)$ gravity has an ill-formulated initial value problem even in vacuo and is regarded as a physically unviable theory.

An alternative approach to the initial value problem is by mapping the equivalent Brans-Dicke theory into its Einstein frame representation, in which the (redefined) scalar degree of freedom couples minimally to gravity but nonminimally to matter [54]. This nonminimal coupling is, of course, absent in vacuo and, from the point of view of the Cauchy problem, plays a very minor role in the presence of matter. In this approach, the non-dynamical role of the scalar is even more obvious, and the conclusions above are reached by using well-known theorems on the initial value problem of general relativity with a scalar field [54].

The problem with Palatini $f(R)$ gravity has been noticed with an entirely different approach, *i.e.*, matching static interior and exterior solutions with spherical symmetry [56] (other problems are reported in Refs. [7, 57, 58]).

The field equations of Palatini $f(R)$ gravity are second order PDEs in the metric. Because f is a function of $\tilde{R}$, which is an algebraic function of T due to eq. (2.12), the right hand side of eq. (2.15) includes second derivatives of T. But T contains derivatives of the matter fields up to first order, hence eq. (2.15) contains derivatives of the matter fields up to third order.

This is in contrast with the situation of general relativity and most of its extensions, in which the field equations contain only first order derivatives of the matter fields. As a consequence, in these theories the metric is generated by an integral over the matter sources and, therefore, discontinuities (or singularities) in the matter fields and their derivatives do not imply unphysical discontinuities of the metric. In Palatini $f(R)$ gravity, instead, the algebraic dependence of the metric on the matter fields creates unacceptable discontinuities in the metric and singularities in the curvature, which is what is found in [56]. So, both the failure of the initial value problem and the occurrence of curvature singularities in the presence of discontinuities in the matter fields or their derivatives can be traced to the fact that the scalar degree of freedom is non-dynamical and is related algebraically to T. A possible cure is to modify the gravitational sector of the action to raise the order of the field equations.

5   Conclusions

We are now ready to summarize the situation of $f(R)$ gravity. Let us stress once again that we regard these theories more as toy models, and as proofs of principle that modified gravity can explain the observed acceleration of the universe without dark energy, than definitive theories.

- **Metric $f(R)$ gravity:** models exist that pass all the observational and theoretical constraints. An example is the Starobinsky model [59]

$$f(R) = R + \lambda R_0 \left[\frac{1}{\left(1 + \frac{R^2}{R_0^2}\right)^n} - 1 \right] .$$

(5.1)

  All the viable models require the chameleon mechanism in order to pass the weak-field limit tests. A condition that must be satisfied by all metric $f(R)$ theories in order to avoid the Dolgov-Kawasaki local instability is $f''(R) \geq 0$. The condition (4.12) must be satisfied for the stability of a de Sitter space. The biggest problem is whether curvature singularities exist for relativistic strong field stars.

- **Palatini $f(R)$ gravity:** these theories suffered multiple deaths; they contain a non-dynamical scalar field, the Cauchy problem is ill-posed, and discontinuities in the matter distribution generate curvature singularities.

- **Metric-affine gravity:** this class of theories is not yet sufficiently developed to assess whether it is viable according to the criteria listed here, and its cosmological consequences are unexplored.

In conclusion, $f(R)$ theories have helped our understanding of the peculiarities of general relativity in the broader spectrum of relativistic theories of gravity, and have taught us about important aspects of its simple generalizations. They even constitute viable alternatives to dark energy models in explaining the cosmic acceleration, although at present there is no definite prediction that sets them apart once and for all from dark energy and other models.

Acknowledgments

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