Supplement to Chapter 14:
The Construction of a Freely Falling Frame

This supplement extends the discussion in Section 14.6 on how to construct the coordinates of a freely falling frame by using gyroscopes to define the directions of three spatial axes. Recall the general idea: An observer in a freely falling laboratory initially orients gyroscopes in three orthogonal directions. They remain orthogonal as a consequence of (14.6) (Problem 1). Three unit vectors along the directions of the spins $\mathbf{e}_1(\tau)$, $\mathbf{e}_2(\tau)$, and $\mathbf{e}_3(\tau)$ together with the four-velocity of the laboratory $\mathbf{u}(\tau) \equiv \mathbf{e}_0(\tau)$ constitute an orthonormal basis for each point labeled by τ along the geodesic. (Although the basis is orthonormal we don't use hats over the indices because this basis is also going to be the coordinate basis for the freely falling frame.) To use these vectors to construct the coordinates of a freely falling frame proceed as follows: Let the proper time of the observer be one coordinate, and let the observer's geodesic be the origin of the three spatial coordinates. At the point labeled by τ along the observer's geodesic pick a unit vector $\mathbf{n}$ with components $(0, n^1, n^2, n^3)$. Follow a spatial geodesic in the direction of $\mathbf{n}$ a distance s from the point on the observer's geodesic. Assign the point which is reached the coordinates

$$x^\alpha = (x^0, x^1, x^2, x^3) \equiv (\tau, sn^1, sn^2, sn^3) \ . \tag{1}$$

Repeat for different directions $\mathbf{n}$ at each τ, and for all τ along the observer's geodesic. In this way a coordinate system labeling each point in the neighborhood of the observer's geodesic can be defined. The components of the unit vector $\mathbf{n}$ can be parametized by polar angles according to $n^i = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$ ensuring the vector is of unit length. The x^i are related to s and the polar angles like usual Cartesian coordinates are related to r and polar angles. We now show that these are the coordinates of a freely falling frame in which the Christoffel symbols vanish on the observer's geodesic.

The demonstration makes use of the geodesic equation (14.3), the gyroscope equation (14.6), and the equation for the spatial geodesics

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\,\frac{dx^\gamma}{ds} = 0 \ . \tag{2}$$

The key observation is that the derivatives $d^2x^\alpha/d\tau^2$ and d^2x^α/ds^2 all vanish because of (1) which is linear in both τ and s. Further, since the coordinate axes were defined to lie along the directions of the spins, the components of each spin are constant, e.g. for the first gyro $s^\alpha = (0, s, 0, 0)$ where s is the magnitude of the spin. Thus $ds^\alpha / d\tau = 0$ for each spin. Now let's see the consequences of these facts.

In these coordinates, where $u^\alpha = (1, 0, 0, 0)$, we have

$$\Gamma^\alpha_{\tau\tau} = 0 \tag{3}$$

from the observer's geodesic equation (14.3). Similarly from the equation for spatial geodesics (2)

$$\Gamma^\alpha_{\beta\gamma} n^\beta n^\gamma = 0 \; . \tag{4}$$

But since the the n^i can point in any direction this implies that

$$\Gamma^\alpha_{ij} = 0 \tag{5}$$

on the observer's geodesic. There remain the Christoffel symbols of the form $\Gamma^\alpha_{\tau i}$. However since the components of the spins are constant in these coordinates the three spatial gyroscope equations (14.6) imply

$$\Gamma^\alpha_{\tau i} = 0 \; . \tag{6}$$

Thus all the Christoffel symbols vanish on the observer's geodesic and the x^α defined by (1) are the coordinates of a freely falling frame.