Supplement to Chapter 22: Stress-Energy Tensor for Short-Wavelength, Linearized Gravitational Waves

In this supplement we return to justify the expression (16.22) for the energy flux in a short-wavelength linearized gravitational wave. In the process we will also a derive an effective stress-energy tensor for such waves.

As discussed in Section 16.5, the equivalence principle prohibits any notion of the density of gravitational energy at a point. However, when spacetime can be characterized as a small ripple in curvature propagating in a smooth background geometry, then there is an *approximate* notion of energy density of the gravitational wave. This energy density is not exactly local. It is an average energy density over spacetime volumes whose dimensions are larger than the wavelength of the wave. We will now derive this expression.

When we solved the Einstein equation for the linearized gravitational wave in Chapter 16, we did not worry about the energy in the wave producing additional spacetime curvature. That was a consistent approximation because we were solving the Einstein equation to first (linear) order in the amplitude of the wave. We expect the energy density in the wave to be *second* order in the amplitude of the wave as (16.21) illustrates. The energy density in the wave could be consistently neglected at linear order, but the stress-energy in those waves becomes a source of curvature in the next order of approximation. By writing out the vacuum Einstein equation in this next order of approximation we will be able to identify the effective stress-energy of linearized gravitational waves.

Einstein's equation for the metric $g_{\alpha\beta}(x)$ in the absence of other sources is

$$R_{\alpha\beta}(g) = 0 \tag{1}$$

to any order of approximation. (In this notation $R_{\alpha\beta}(g)$ means that the components of $R_{\alpha\beta}$ depend on all 10 metric components $g_{00}(x),\ g_{01}(x),\cdots$.) Write the metric as

$$g_{\alpha\beta}(x) = \gamma_{\alpha\beta}(x) + h_{\alpha\beta}(x) \tag{2}$$

where $\gamma_{\alpha\beta}$ is a smooth *background metric* (B) and $h_{\alpha\beta}$ represents a small amplitude propagating ripple of curvature whose wavelength λ is much smaller than the scale $\mathcal{R}$ on which background metric varies significantly. The idea is that $\gamma_{\alpha\beta}$ will be close to flat, but slightly curved due to the energy in the gravitational wave.

Expand the Einstein equation (1) in powers of the amplitude of the wave, obtaining

$$R_{\alpha\beta} = R^{(B)}_{\alpha\beta}(\gamma) + R^{(1)}_{\alpha\beta}(\gamma, h) + R^{(2)}_{\alpha\beta}(\gamma, h) + \cdots = 0 \ . \tag{3}$$

Here, $R^{(B)}_{\alpha\beta}$ is the smooth background curvature independent of $h_{\alpha\beta}$, $R^{(1)}_{\alpha\beta}$ is linear in $h_{\alpha\beta}$, and $R^{(2)}_{\alpha\beta}$ is quadratic in $h_{\alpha\beta}$. We will not need the higher order terms.

If the background metric is curved only by the energy in the waves, that curvature cannot be linear in $h_{\alpha\beta}$; the energy is quadratic. Therefore, although the background and quadratic term may be comparable in (3), the linear part must vanish by itself

$$R^{(1)}_{\alpha\beta}(\gamma, h) = 0 \ . \tag{4}$$

This is a linear wave equation for the gravitational wave. The smaller the amplitude of the wave, the less curved the background metric $\gamma_{\alpha\beta}$, and the closer (4) comes to the linear wave equation in flat space derived in Chapter 16.

Short-wavelength waves vary rapidly on the scales that $\gamma_{\alpha\beta}$ varies; that's what's meant by *short*-wavelength. The remainder of equation (3) can therefore hold only in an average sense. Averaging it over a spacetime volume with sides larger than a wavelength and denoting such an average by $\langle\cdot\rangle$ we obtain

$$R^{(B)}_{\alpha\beta}(\gamma) = -\left\langle R^{(2)}_{\alpha\beta}(\gamma, h)\right\rangle \ . \tag{5}$$

Equation (5) shows how quadratic terms in the amplitude of the gravitational wave generate spacetime curvature. We have thus identified the effective stress-energy of linearized gravitational waves. By writing (5) in the form of an Einstein equation for the background curvature

$$R^{(B)}_{\alpha\beta}(\gamma) = 8\pi\left(T^{(GW)}_{\alpha\beta} - \frac{1}{2}\gamma_{\alpha\beta}T^{(GW)}\right) \tag{6}$$

we make the identification explicit. The the effective gravitational wave stress-energy is

$$T^{(GW)}_{\alpha\beta} = -\frac{1}{8\pi}\left[\left\langle R^{(2)}_{\alpha\beta}\right\rangle - \frac{1}{2}\gamma_{\alpha\beta}\left\langle R^{(2)}\right\rangle\right] \ . \tag{7}$$

Energy Flux in a Plane Wave

The effective short-wavelength gravitational wave stress-energy (7) is the origin of the expression (16.22) for the energy flux in a plane linearized gravitational wave to lowest order in its amplitude. To see this let's evaluate (7) for the plane wave (16.2). This metric represents a $+$ polarized plane wave propagating in the flat spacetime in the $z-$direction. To make contact with the expression (16.22) for the energy flux we assume the wave has a definite frequency ω and amplitude a as in so that $f(t-z) = a\sin[\omega(t-z)]$. The energy flux is the component $T^{tz}_{(GW)}$ [cf. (22.19)]. We now evaluate this to the leading order in a which is a^2.

To calculate $T^{tz}_{(GW)}$ to quadratic order in the amplitude a it is enough to evaluate (7) with $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$. The resulting stress-energy causes $\gamma_{\alpha\beta}$ to deviate from $\eta_{\alpha\beta}$ by an amount proportional to a^2 which could be found by solving (6). But including that correction to $\gamma_{\alpha\beta}$ would not affect the value $T^{tz}_{(GW)}$ to quadratic order in a^2, because (7) is already quadratic in $h_{\alpha\beta}$

To evaluate (7) with (16.22) note that the Ricci curvature is given in terms of the Christoffel symbols by (21.32), the Christoffel symbols are given in terms of the metric by (20.53), and the metric is given by (16.2). To calculate $R^{(2)}_{tz}$, we need only calculate R_{tz} to the second order in the amplitude of the wave $h_{\alpha\beta}$ with $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$ as described above. This is a straightforward if tedious enterprise. We summarize just enough of the intermediate steps in the calculation of $T^{tz}_{(GW)}$ to lead the reader through it for the case of the wave (16.2) with frequency ω. That wave $h_{\alpha\beta}$ has no components except in the transverse x- and y-directions. This is enough to show $\Gamma^{\alpha}_{tz} = \Gamma^{\alpha}_{tt} = \Gamma^{\alpha}_{zz} = 0$. That simplifies the form of R_{tz}. The only other novelty here is that the calculation of the Christoffel symbol to second order in $h_{\alpha\beta}$ involves knowing the first order change in the inverse metric $g^{\alpha\beta}$. However, the basic relation $g^{\alpha\gamma}g_{\alpha\beta} = \delta^{\alpha}_{\gamma}$ is enough to establish that to first order in $h_{\alpha\beta}$

$$g^{\alpha\beta}_{(1)} = -h^{\alpha\beta} \tag{8}$$

where the indices on $h^{\alpha\beta}$ are raised with $\eta^{\alpha\beta}$. Putting all this together, one finds:

$$R_{tz} = \frac{1}{4}\frac{\partial h_{ij}}{\partial t}\ \frac{\partial h_{ij}}{\partial z} + \frac{1}{2}\, h^{ij}\ \frac{\partial^2 h_{ij}}{\partial t \partial z}\ . \tag{9}$$

Putting in the form of the wave (16.2), averaging over a period, and raising indices using $\eta^{\alpha\beta}$, one finds

$$T^{tz}_{(GW)} = \frac{\omega^2}{32\pi} \left\langle h_{ij} h^{ij} \right\rangle , \tag{10a}$$

$$= \frac{\omega^2 a^2}{32\pi} . \tag{10b}$$

In arriving at these expressions use was made of the fact that the average of both $\sin^2(\omega t)$ and $\cos^2(\omega t)$ over a period is $1/2$ and that there are two, equal magnitude metric components in TT-gauge for a gravitational wave. Thus we derive the expression (16.22) for the energy flux in a linearized gravitational plane wave with definite frequency and $+$ or $-$ polarization.

The General Stress Energy

The same methods that were used to evaluate $T^{tz}_{(GW)}$ can be used to evaluate all the components of $T^{(GW)}_{\alpha\beta}$ from (7). The calculations are equally straightforward, just a little more tedious. (For the details see, for instance, Misner, Thorne, and Wheeler 1973.) Assuming a flat background $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$ and usual rectangular coordinates (t,x,y,z), the complete result is

$$\boxed{T^{(GW)}_{\alpha\beta} = \frac{1}{32\pi} \left\langle \partial_\alpha h^{TT}_{ij} \partial_\beta \, h^{ij}_{TT} \right\rangle} \tag{11}$$

where the wave components are assumed to be in transverse traceless gauge. This is called the *Isaacson* stress-energy tensor. It is easy to check that (11) reproduces (10b) for a wave of the form (16.2). Eqs (11) and (6) demonstrate that gravitational waves carry energy and are themselves a source of curvature. That reflects the non-linear nature of the Einstein equation.