Physics 130 General Relativity Seminar

Assignment 10 April 01, 2013

General topic: **The Einstein Equation**

Part 1: Readings

Hartle: Chap 20 A Little More math

Hartle: Chap 21- Sections 1-4 Curvature and Einstein's Equation

Hartle: Chap 21 Supplement - Deriving Equation of Geodesic Deviation and Formula for Riemann Tensor

Part 2: Problems Hartle Problems

1. Hartle 20.04 Coordinate basis components of a gradient
2. Hartle 20.05 The upstairs basis
3. Hartle 20.07 Basis and dual vectors
4. Hartle 20.19 Normal vector and null curves
5. Hartle 20.25 Parallel propagation
6. Hartle 20.26 Connected by a Lorentz boost
7. Hartle 21.10 Uniform gravitational field
8. Hartle 21.12 Curvature of the wormhole

Boccio Extra Problems

1. **A Two-Dimensional World**

   A certain two-dimensional world is described by the metric

$$ds^2 = \frac{dx^2 + dy^2}{\left[1 + \frac{x^2+y^2}{4a^2}\right]^2}$$

   (a) Compute the connection coefficients Γ^i_{jk}

   (b) Let $\vec{\xi} = -y\hat{e}_x + x\hat{e}_y$. Show that $\vec{\xi}$ is a solution of Killings equation.

(c) What is the conserved quantity that corresponds to this symmetry? Show from the geodesic equation that this quantity is indeed conserved.

(d) Compute the Riemann tensor R^{ij}_{kl}, the Ricci tensor R^i_j, and the Ricci scalar R. What is the shape of this world?

2. **More Geodesics**

   Consider the 2-dimensional metric

$$ds^2 = a^2 \left(d\chi^2 + \sinh^2 \chi d\varphi^2 \right)$$

   (a) Compute the connection coefficients Γ^i_{jk}

   (b) Compute all components of the Riemann tensor R^{ij}_{kl}, the Ricci tensor R^i_j, and the Ricci scalar R.

   (c) A geodesic starts at $\chi = b$, $\varphi = 0$ with tangent $d\varphi/d\lambda = 1$, $d\chi/d\lambda = 0$. Find the trajectory $\chi(\varphi)$.

   (d) A second geodesic starts at $\chi = b + \xi (\xi \ll 1)$, also initially in the φ-direction. How does the separation initially increase or decrease along the two curves.

   (e) What is the shape of the geodesic trajectory as $a \to \infty$, $\chi \to 0$ with $r = a\chi$ fixed.

3. **Parallel Transport on a Sphere**

   On the surface of a 2−sphere of radius a

$$ds^2 = a^2 \left(d\theta^2 + \sin^2 \theta d\varphi^2 \right)$$

   Consider the vector $\vec{A}_0 = \vec{e}_\theta$ at $\theta = \theta_0$, $\varphi = 0$. The vector is parallel transported all the way around the latitude circle $\theta = \theta_0$ (i.e., over the range $0 \le \varphi \le 2\pi$ at $\theta = \theta_0$). What is the resulting vector $\vec{A}$? What is its magnitude $\left(\vec{A} \cdot \vec{A} \right)^{1/2}$? HINT: derive differential equations for A^θ and A^φ as functions of φ.

4. **Curvature on a Sphere**

   (a) Compute all the nonvanishing components of the Riemann tensor R_{ijkl} $((i, j, k, l) \in (\theta, \varphi))$ for the surface of a 2−sphere.

(b) Consider the parallel transport of a tangent vector $\vec{A} = A^\theta \hat{e}_\theta + A^\varphi \hat{e}_\varphi$ on the sphere around an infinitesimal parallelogram of sides $\hat{e}_\theta d\theta$ and $\hat{e}_\varphi d\varphi$. Using the results of part (a), show that to first order in $d\Omega = \sin\theta d\theta d\varphi$, the length of $\vec{A}$ is unchanged, but its direction rotates through an angle equal to $d\Omega$.

(c) Show that, if $\vec{A}$ is parallel transported around the boundary of any simply connected solid angle Ω, its direction rotates through an angle Ω. (*Simply connected* is a topological term meaning that the boundary of the region could be shrunk to a point; it tells us that there are no holes in the manifold or other pathologies). Using the result of part (b) and intuition from proofs of Stokes theorem, this should be an easy calculation. Compare with the result of EP #32.

5. **Riemann Tensor for 1+1 Spacetimes**

(a) Compute all the nonvanishing components of the Riemann tensor for the spacetime with line element

$$ds^2 = -e^{2\varphi(x)}dt^2 + e^{2\psi(x)}dx^2$$

(b) For the case $\varphi = \psi = \frac{1}{2}\ln|g(x - x_0)|$ where g and x_0 are constants, show that the spacetime is flat and find a coordinate transformation to globally flat coordinates $(\bar{t}, \bar{x})$ such that $ds^2 = -d\bar{t}^2 + d\bar{x}^2$.

6. **About Vectors Tangent to Geodesics**

Let $x^\mu(\tau)$ represent a timelike geodesic curve in spacetime, where τ is the proper time as measured along the curve. Then $u^\mu \equiv dx^\mu/d\tau$ is tangent to the geodesic curve at any point along the curve.

(a) If $g_{\mu\nu}$ is the metric of spacetime, compute the magnitude of the vector u^μ. Do not use units where $c = 1$, but keep any factors of c explicit. Compare your result with the one obtained in flat Minkowski spacetime. HINT: The magnitude of a timelike vector v^μ is given by $(-g_{\mu\nu}v^\mu v^\nu)^{1/2}$.

(b) Consider a contravariant timelike vector v^μ at a point P on the geodesic curve. Move the vector v^μ from the point P to an arbitrary point Q on the geodesic curve via parallel transport. Prove that the magnitude of the vector v^μ at the point Q equals the magnitude of the vector v^μ at point P.

(c) Suppose that at the point P on the geodesic curve, $v^\mu = u^\mu$. Now, parallel transport the vector v^μ along the geodesic curve to arbitrary point Q. Show that $v^\mu = u^\mu$ at the point Q. NOTE: This result implies that a vector tangent to a geodesic at a given point will always remain tangent to the geodesic curve when parallel transported along the geodesic.