Physics 130 General Relativity Seminar

Assignment 5 February 18, 2013

General topic: **Curved Spacetimes in General Relativity**

Part 1: Readings

Hartle: Ch 9 - Geometry Outside Spherical Star

Part 2: Problems Hartle Problems

1. Hartle 9.06 Falling inwards
2. Hartle 9.07 How much faster?
3. Hartle 9.08 What is the period?
4. Hartle 9.09 Rate of change of angular position
5. Hartle 9.11 Perturb a circular orbit
6. Hartle 9.12 Comet speed
7. Hartle 9.14 Kepler's area law
8. Hartle 9.15 Perihelion precession
9. Hartle 9.18 Deflection of light in a different geometry
10. Hartle 9.20 Unstable circular orbits

Boccio Extra Problems

1. **Velocity of Light**

   The Schwarzschild metric is given by

   $$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

   As a function of r, what is the coordinate velocity of light in this metric (a) in the radial direction? (b) in the transverse direction? What are the physical consequences of these results.

2. **Orbiting Photons**

   Consider a photon in orbit in a Schwarzschild geometry. For simplicity, assume that the orbit lies in the equatorial plane (i.e., $\theta = \pi/2$ is constant).

(a) Show that the geodesic equations imply that

$$\bar{E}^2 = \frac{1}{c^2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{\bar{J}^2}{c^2 r^2}\left(1 - \frac{2GM}{c^2 r}\right)$$

where $\bar{E}$ and $\bar{J}$ are constant of the motion and λ is an affine parameter.

(b) Define the effective potential

$$V_{eff} = \frac{\bar{J}^2}{c^2 r^2}\left(1 - \frac{2GM}{c^2 r}\right)$$

The effective potential yields information about the orbits of massive particles. Show that for photons there exists an unstable circular orbit of radius $3r_s/2$, where $r_s = 2GM/c^2$ is the Schwarzschild radius. HINT: Make sure you check for minima and maxima.

(c) Compute the proper time for the photon to complete one revolution of the circular orbit as measured by an observer stationed at $r = 3r_s/2$.

(d) What orbital period does a very distant observer assign to the photon?

(e) The instability of the orbit can be exhibited directly. Show, by perturbing the geodesic in the equatorial plane, that the circular orbit of the photon at $r = 3r_s/2$ is unstable. HINT: in the orbit equation put $r = 3r_s/2 + \eta$, and deduce an equation for η. Keep only the first order terms in $\eta << 1$, and solve the resulting equation.

3. **Circular Orbit**

An object moves in a circular orbit at Schwarzschild radius R around a spherically symmetric mass M. Show that the proper time τ is related to coordinate time t by

$$\frac{\tau}{t} = \sqrt{1 - \frac{3M}{R}}$$

HINT: It is helpful to derive a relativistic version of Keplers third law.

4. **Astronauts in Orbit**

Consider a spacecraft in a circular orbit in a Scharzschild geometry. As usual, we denote the Schwarzschild coordinates by (ct, r, θ, φ) and assume that the orbit occurs in the plane where $\theta = \pi/2$. We denote two conserved quantities by

$$e = \left[1 - \frac{r_s}{r}\right] \frac{dt}{d\tau} \quad and \quad \ell = r^2 \frac{d\varphi}{d\tau}$$

where $r_s = 2GM/c^2$ and τ is the proper time.

(a) Write down the geodesic equation for the variable r. Noting that r is independent of τ for a circular orbit, show that:

$$\frac{\ell}{c} = c\left(\tfrac{1}{2} r_s r\right)^{1/2} \left[1 - \frac{r_s}{r}\right]^{-1}$$

(b) Show that for a timelike geodesic, $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -c^2$, where $\dot{x}^\mu = dx^\mu/d\tau$. From this result, derive a second relation between ℓ and e for a circular orbit. Then, using the result of part (a) to eliminate e, obtain an expression for $d\tau/d\varphi$ in terms of the radius r of the orbit.

(c) Using the result of part (b), determine the period of the orbit as measured by an observer at rest inside the orbiting spacecraft, as a function of the radius r of the orbit.

(d) Suppose an astronaut leaves the spacecraft and uses a rocket-pack to maintain a fixed position at radial distance r equal to the orbital radius and at fixed $\theta = \pi/2$ and $\varphi = 0$. The astronaut outside then measures the time it takes the spacecraft to make one orbital revolution. Evaluate the period as measured by the outside astronaut. Does the astronaut outside the spacecraft age faster or slower than the astronaut orbiting inside the spacecraft?