Partial Differential Equations (PDEs)

In Physics, there are differential equations of motion that describe the response of systems to external disturbances. These are called ordinary differential equations (ODEs). There are also differential equations of states, or field equations, whose solutions give the space-time dependence of physical properties. These are called partial differential equations (PDEs) in the 4 variables x,y,z,t.

In general, the PDE's we will discuss describe three-dimensional situations. The independent variables are the position vector $\vec{r}$ and the time t. The actual variables used to specify $\vec{r}$ are dictated by the coordinate system in use, i.e., (x,y,z), (ρ,ϕ,z), (r,θ,ϕ), etc.

The most important PDEs are:

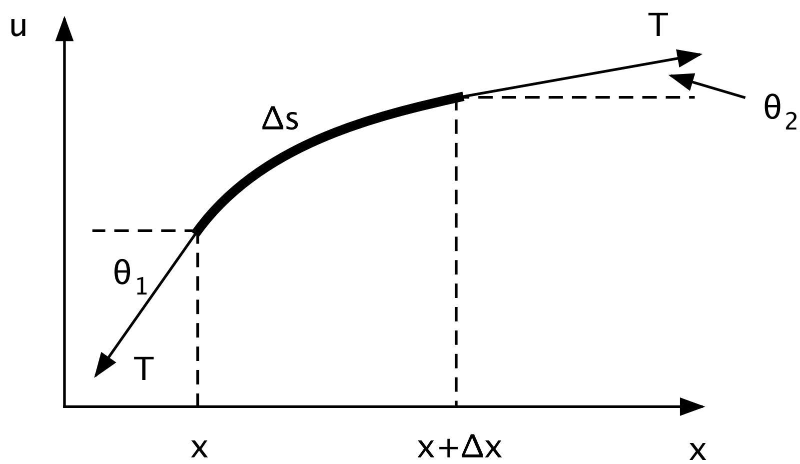
(1) The wave equation

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

This equation describes as a function of position and time the displacement from equilibrium, $u(\vec{r},t)$, of a vibrating string or membrane, or a vibrating solid, gas or liquid. The equation also occurs in electromagnetism, where $u(\vec{r},t)$ may be a component of the electric or magnetic field in an electromagnetic wave, or the current or voltage along a transmission line. The quantity c is the speed of propagation of the waves.

We now derive the wave equation in a special case. We consider the small transverse displacements $u(x,t)$ of a uniform string of mass per unit length ρ held under uniform tension T, assuming that the string is initially located along the x-axis in a Cartesian coordinate system.

The figure below shows the forces acting on an elemental length Δs of the string.



If the tension T in the string is uniform along its length, the net upward vertical force on the element is

$$\Delta F = T\sin\theta_2 - T\sin\theta_1$$

Assuming that the angles θ_1 and θ_2 are both small, we may make the approximation $\sin\theta \approx \tan\theta$. Since, at any point on the string the slope is

$$\tan\theta = \frac{\partial u}{\partial x}$$

the force can be written as

$$\Delta F = T\left(\frac{\partial u(x+\Delta x,t)}{\partial x} - \frac{\partial u(x,t)}{\partial x}\right) \approx T\,\frac{\partial^2 u(x,t)}{\partial x^2}\Delta x$$

where we have used the standard definition of the partial derivative.

The upward force may be equated, by Newton's second law, to the product of the mass of the element and its upward acceleration. The element has mass $\rho\Delta s$, which is approximately equal to $\rho\Delta x$ if the vibrations of the string are small, and so we have

$$\rho\Delta x\frac{\partial^2 u(x,t)}{\partial t^2} = T\,\frac{\partial^2 u(x,t)}{\partial x^2}\Delta x$$

$$\rightarrow \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2} \quad , \quad c^2 = \frac{T}{\rho}$$

which is the one-dimensional form of the wave equation.

(2) The diffusion equation

$$\nabla^2 u = \frac{1}{D}\frac{\partial u}{\partial t}$$

This equation describes the temperature $u(\vec{r},t)$ in a region containing no heat sources or sinks. It also applies to the diffusion of a chemical that has concentration $u(\vec{r},t)$. The constant D is called the diffusivity.

We now derive now derive the diffusion equation satisfied by the temperature $u(\vec{r},t)$ at time t for a material of uniform thermal conductivity k, specific heat capacity s, and density ρ.

Let us consider an arbitrary volume V lying within a solid, and bounded by a surface S. At any point in the solid the rate of heat flow per unit area in any given direction $\vec{r}$ is proportional to minus the component of the temperature gradient in that direction and is given by

$$(-k\nabla u)\cdot\vec{r}$$

The total flux of heat **out** of the volume V per unit time is given by

$$-\frac{dQ}{dt} = \iint_S (-k\nabla u)\cdot\hat{n}dS = \iiint_V \nabla\cdot(-k\nabla u)dV$$

where Q is the total heat energy in V at time t, and $\hat{n}$ is the outward-pointing unit normal to S; note that we have used the divergence theorem to convert the surface integral into a volume integral.

We can also express Q as the volume integral over V,

$$Q = \iiint_V s\rho u dV$$

and so its rate of change is given by

$$\frac{dQ}{dt} = \iiint_V s\rho\frac{\partial u}{\partial t}dV$$

Comparing the two expressions for dQ/dt and remembering the volume V is arbitrary, we okbtain the three-dimensional diffusion equation

$$\nabla^2 u = \frac{1}{D}\frac{\partial u}{\partial t} \quad , \quad D = \frac{k}{s\rho}$$

(3) Laplace's equation

$$\nabla^2 u = 0$$

This equation is obtained by setting

$$\frac{\partial u}{\partial t} = 0$$

in the diffusion equation and describes (for example) the **steady-state** temperature of a solid in which there are no heat sources - i.e., the temperature after a long time has elapsed.

Laplace's equation also describes the gravitational potential in a region containing no matter, or the electrostatic potential in a charge-free region. It also applies to the flow of an incompressible fluid with no sources, sinks or vortices - in this case $u(\vec{r},t)$ is the velocity potential, from which the velocity is given by $\vec{v} = \nabla u$.

(4) Poisson's equation

$$\nabla^2 u = \rho(\vec{r})$$

This equation describes the same physical situations as Laplace's equation, but in regions containing matter, charges, or sources of heat or fluid. The function $\rho(\vec{r})$ is called the source density, and in physical applications usually contains some multiplicative physical constants. For example, if u is the electrostatic potential in some region of space, in which $\rho(\vec{r})$ is the density of electric charge, then

$$\nabla^2 u = -\frac{1}{\varepsilon_0}\rho(\vec{r})$$

where ε_0 is the permittivity of free space. Alternatively, u might represent the gravitational potential in some region where the matter density is given by $\rho(\vec{r})$; then

$$\nabla^2 u = 4\pi G \rho(\vec{r})$$

where G is the gravitational constant.

(5) Schrodinger equation

$$-\frac{\hbar^2}{2m}\nabla^2 u + V(\vec{r})u = i\hbar\frac{\partial u}{\partial t}$$

This equation describes the quantum mechanical wave function $u(\vec{r},t)$ of a non-relativistic particle of mass m; $\hbar$ is Planck's constant divided by 2π.

All of these equations are **linear**, They are all 2nd-order in the space variables and of 1st- or 2nd-order in time.

The use of these differential operators guarantees several things:

(1) differential operators imply invariance with respect to space and time translations and hence conservation of energy and momentum

(2) the differential operator ∇^2 is the simplest operator that will be invariant under the parity transformation (inversion)

(3) equations that are 2nd-order in time are invariant under time reversal ($t \to -t$) and hence, a movie of the system in time should represent a real physical system whether it is run forwards or backwards. For example, the wave equation might have a solution representing a wave propagating to the right and if we run the movie backwards, we get a wave propagating to the left, which is also a valid solution.

In the diffusion or heat conduction, the field equation (for the density or temperature fields) is only 1st-order in time. The equation does not, and should not, satisfy time-reversal invariance, since heat is known to flow from high temperature to low temperature and NEVER the other way around. A movie of a pool of water solidifying into a block of ice on a hot day has obviously been run backwards.

The Schrodinger equation is a sort of diffusion equation with an imaginary diffusion constant; the wave function is a complex function.

The textbook discusses general aspects of PDEs and their solutions. We will concentrate in class on one solution method, namely, **separation of variables.**

Separation of Variables and Eigenfunction Expansions

Under certain circumstances the solution of a PDE may be written as a sum of terms, each of which is the product of functions of only one of the variable. This is called solution by **separation of variables** (**SOV**). Let us illustrate the procedure by an example:

Consider a 1-dimensional wave equation describing the transverse vibrations of a string

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

In the SOV method we simply look for a solution of the form

$$u(x,t) = X(x)T(t)$$

Direct substitution then gives

$$\frac{\partial^2}{\partial x^2}u(x,t) = T(t)\frac{d^2 X(x)}{dx^2} = \frac{1}{c^2}X(x)\frac{d^2 T(t)}{dt^2}$$

$$\frac{1}{X(x)}\frac{d^2 X(x)}{dx^2} = \frac{1}{c^2}\frac{1}{T(t)}\frac{d^2 T(t)}{dt^2}$$

It is clear that we have **separated the variables.** Since the LHS is a function of x only and the RHS is a function of t only, both sides of this equation must be equal for all x and t. The only way this is possible is for both of them to be equal to the same constant, say λ.

$$\frac{1}{X(x)}\frac{d^2 X(x)}{dx^2} = \lambda = \frac{1}{c^2}\frac{1}{T(t)}\frac{d^2 T(t)}{dt^2}$$

which gives

$$\frac{d^2 X(x)}{dx^2} - \lambda X(x) = 0 \quad \text{and} \quad \frac{d^2 T(t)}{dt^2} - \lambda c^2 T(t) = 0$$

These are two **separated** ODEs (single variable). They are **not completely independent** of each other since the **same** separation constant λ must appear in both. They are both eigenvalue/ eigenfunction equations.

The general solutions of these equations are:

$$X(x) = A\cos(\sqrt{-\lambda}x) + B\sin(\sqrt{-\lambda}x)$$

$$T(t) = D\cos(\sqrt{-\lambda}ct) + E\sin(\sqrt{-\lambda}ct)$$

As a rule, **all possible values** of the separation constant λ are allowed unless **explicitly forbidden** by the physics of the system. i.e., certain values of λ can be forbidden when the corresponding solution $X(x)$, which depends on λ, does not have the correct properties. The properties in question, are the **boundary conditions** imposed by the physics of the system.

It may happen that one or more of these boundary conditions can be satisfied only when the separation constant takes on a set of special values. This set then contains the only permissible values, or **eigenvalues** , for the problem. The corresponding solutions are called **eigenfunctions.** Let us illustrate this with a particular example:

Suppose, as in the case of the Fourier series, we are interested in solutions with a period of 2π i.e.,

$$1 \quad , \quad \cos(nx) \quad , \quad \sin(nx) \quad , \quad n > 0 \text{ , integer}$$

This implies that the only permissible constants are

$$\lambda_n^2 = -n^2 \qquad , \quad \text{n=1,2,3,4,.....}$$

For each $\lambda = \lambda_n$ we then get a wave solution of the form

$$\begin{aligned} X_n(x)T_n(t) &= \left(A_n\cos(nx) + B_n\sin(nx)\right)\left(D_n\cos(nct) + E_n\sin(nct)\right) \\ &= a_n\cos(nx)\cos(nct) + b_n\sin(nx)\cos(nct) + d_n\cos(nx)\sin(nct) + e_n\sin(nx)\sin(nct) \end{aligned}$$

Since the 1-dimensional wave equation is linear, the general solution periodic in x with period 2π is then the linear superposition

$$u(x,t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} X_n(t)T_n(t)$$

of all possible solutions. Note that this is a **double Fourier series.**

Boundary and Initial Conditions

The complete determination of a solution of the PDE requires the specification of a suitable set of boundary and initial conditions. The boundaries may not be just points, but, depending on the dimension of the system they can be lines or surfaces.

Let us return to the wave equation. We now specialize the problem and consider the 1-dimensional vibrations of a string **rigidly attached** to a support at the points x = 0 and x = L.

$$\begin{aligned} X_n(x) &= A_n\cos(\sqrt{-\lambda_n}x) + B_n\sin(\sqrt{-\lambda_n}x) \\ &= A_n\cos(k_n x) + B_n\sin(k_n x) \qquad \text{where} \qquad k_n = \sqrt{-\lambda_n} \end{aligned}$$

These boundary conditions mean that

$$X(0) = 0 \rightarrow A_n = 0 \qquad \text{for all } n$$

$$X(L) = 0 \rightarrow B_n \neq 0 \rightarrow \sin(k_n L) = 0 \rightarrow k_n = \frac{n\pi}{L} \qquad \text{for all } n$$

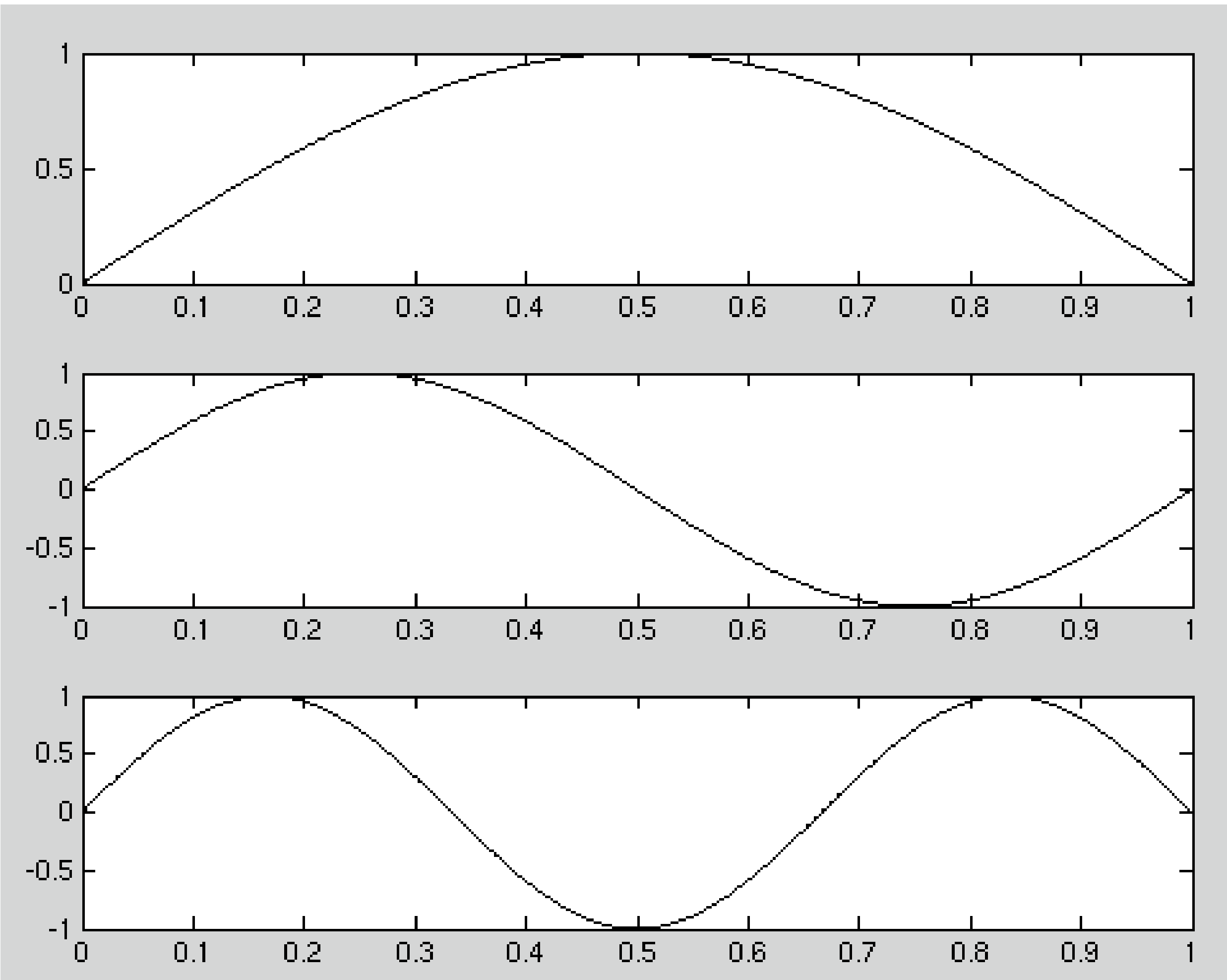
This gives as a solution

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$\lambda_n = -\left(\frac{n\pi}{L}\right)^2 = \text{allowed separation constants}$$

The eigenfunction $X_n(x)$ belongs to the eigenvalue λ_n and describes the nth eigenmode (or normal mode) of the vibration of the string (fixed at both ends)(see figure below)



These represent the **fundamental vibration** along with the **first and second harmonics.**

Note that there are points given by

$$x_m = \frac{mL}{n} \quad \text{m=1,2,3,4,.......,n-1}$$

where the displacement u = 0 or

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right) = B_n \sin(m\pi) = 0$$

which are called **nodal points** of the wave.

The time factor $T_n(t)$ associated with $X_n(t)$ is given by

$$T_n(t) = D_n \cos(\omega_n t) + E_n \sin(\omega_n t)$$

where

$$\omega_n = \frac{nc\pi}{L}$$

is the frequency of vibration of the n^{th} normal mode of the string (fixed at both ends).

Hence, the general wave amplitude function (shape) of the vibrating string fixed at x = 0 and x = L is the general eigenfunction expansion (superposition of all solutions)

$$u(x,t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} X_n(t)T_n(tt) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)(D_n \cos(\omega_n t) + E_n \sin(\omega_n t))$$

If we pluck the string at time t = 0, which mode(s) will be excited?

The answer depends on how we pluck the string or on the **initial conditions** at t = 0.

Since PDE is 2nd order in time, we need 2 initial conditions to completely specify the solution. We usually choose them to be

$$u(x,0) = u_0(x) = \text{initial displacement of the string at } \ t = 0$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = v_0(x) = \text{initial velocity profile of the string at } \ t = 0$$

or we have

$$u(x,0) = u_0(x) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{L}\right)$$
$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = v_0(x) = \sum_{n=1}^{\infty} E_n \omega_n \sin\left(\frac{n\pi x}{L}\right)$$

which implies that

$$D_m = \frac{2}{L}\int_0^L \sin\left(\frac{m\pi x}{L}\right) u_0(x)\,dx$$
$$E_m = \frac{2}{\omega_m L}\int_0^L \sin\left(\frac{m\pi x}{L}\right) v_0(x)\,dx$$

These coefficients determine which normal modes are excited and with what strength.

Real example of a vibrating string(back through everything again):

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

gives

$$y(x,t) = X(x)T(t)$$
$$\frac{X''}{X} = -\alpha^2 = \frac{1}{c^2}\frac{\ddot{T}}{T}$$
$$X'' + \alpha^2 X = 0 \quad , \quad \ddot{T} + \alpha^2 c^2 T = 0$$

Define

$$\alpha c = \omega = 2\pi\nu = \frac{2\pi c}{\lambda} \to \alpha = \frac{2\pi}{\lambda} = k = \text{wave number}$$

The solution is (as before)

$$y = \left(a\sin(kx) + b\cos(kx)\right)\left(d\sin(\omega t) + e\cos(\omega t)\right)$$

The string is fastened at x = 0 and x = L , so that y(0) = y(L) = 0, which gives

$$y(0) = b\left(d\sin(\omega t) + e\cos(\omega t)\right) = 0 \to b = 0$$
$$y(L) = a\sin(kL)\left(d\sin(\omega t) + e\cos(\omega t)\right) = 0 \to \sin(kL) = 0 \to kL = n\pi$$

and

$$y_n = a_n \sin\left(\frac{n\pi x}{L}\right)\left(d\sin\left(\frac{n\pi v t}{L}\right) + e\cos\left(\frac{n\pi v t}{L}\right)\right)$$

Now choose initial conditions at t = 0

$$y(x,0) = f(x) = \begin{cases} x/2 & 0 \le \mathrm{x} \le \mathrm{L/2} \\ L/2 - x/2 & \mathrm{L/2} \le \mathrm{x} \le \mathrm{L} \end{cases}$$

and

$$\left.\frac{\partial y(x,t)}{\partial t}\right|_{t=0} = 0$$

The condition

$$\left.\frac{\partial y(x,t)}{\partial t}\right|_{t=0} = 0 \rightarrow a_n \sin\left(\frac{n\pi x}{L}\right)\frac{n\pi v}{L}\left(d_n \cos\left(\frac{n\pi vt}{L}\right) - e_n \sin\left(\frac{n\pi vt}{L}\right)\right)\Bigg|_{t=0}$$

$$0 = a_n \sin\left(\frac{n\pi x}{L}\right)\frac{n\pi v}{L}\left(d_n\right)\Big| \rightarrow d_n = 0$$

This means we physically pull the string into a triangular shape and let is go from rest. The most general solution is then is

$$y(x,t) = \left(\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\right)\cos\left(\frac{n\pi v}{L}t\right)$$

Now using the other initial conditon, we have

$$y(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

or the A_n are the Fourier coefficients of the sine series for the triangular pulse. We have

$$\begin{aligned}
A_m &= \frac{2}{L}\int_0^L \sin\left(\frac{m\pi x}{L}\right) f(x)dx = \frac{1}{L}\int_0^{L/2} \sin\left(\frac{m\pi x}{L}\right) x dx + \frac{1}{L}\int_{L/2}^{L} \sin\left(\frac{m\pi x}{L}\right)(L-x)dx \\
&= \frac{1}{L}\frac{L^2}{m^2\pi^2}\int_0^{m\pi/2} y \sin y dy - \frac{1}{L}\frac{L^2}{m^2\pi^2}\int_{m\pi/2}^{m\pi} y\sin y dy + \frac{L}{m\pi}\int_{m\pi/2}^{m\pi}\sin y dy \\
&= \frac{L}{m^2\pi^2}(\sin y - y\cos y)\Big|_0^{m\pi/2} - \frac{L}{m^2\pi^2}(\sin y - y\cos y)\Big|_{m\pi/2}^{m\pi} - \frac{L}{m\pi}(\cos y)\Big|_{m\pi/2}^{m\pi} \\
&= \frac{L}{m^2\pi^2}\left(\sin\left(\frac{m\pi}{2}\right) - \frac{m\pi}{2}\cos\left(\frac{m\pi}{2}\right) - \sin(0) + (0)\cos(0)\right) \\
&\quad - \frac{L}{m^2\pi^2}\left(\sin(m\pi) - m\pi\cos(m\pi) - \sin\left(\frac{m\pi}{2}\right) + \frac{m\pi}{2}\cos\left(\frac{m\pi}{2}\right)\right) \\
&\quad - \frac{L}{m\pi}\left(\cos(m\pi) - \cos\left(\frac{m\pi}{2}\right)\right)
\end{aligned}$$

or

$$A_m = \begin{cases} \left(\dfrac{2L}{m^2\pi^2}\right)\sin\left(\dfrac{m\pi}{2}\right) & \text{m odd} \\ 0 & \text{m even} \end{cases}$$

and so on.

The other possible initial condition is

$$y(x,0)=0$$

and

$$\left.\frac{\partial y(x,t)}{\partial t}\right|_{t=0}=f(x)=\begin{cases} x/2 & 0 \le \mathrm{x} \le \mathrm{L/2} \\ L/2-x/2 & \mathrm{L/2} \le \mathrm{x} \le \mathrm{L} \end{cases}$$

which corresponds to hitting the string when it is flat. This gives

$$y(x,t)=\left(\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\right)\sin\left(\frac{n\pi v}{L}t\right)$$

$$\left.\frac{\partial y(x,t)}{\partial t}\right|_{t=0}=f(x)=\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)=\text{initial velocity profile}$$

For each n (in either case) there is a different frequency

$$f_n=\frac{\omega_n}{2\pi}=\frac{n\pi v}{2\pi L}=\frac{nv}{2L}$$

which are the **normal mode frequencies.** These frequencies **can be excited separately and would be stable.** The string would vibrate in a single mode or one term of the general sum

$$\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi v}{L}t\right)$$

If we took a photograph at any given value of t, we get a picture of the string

$$y=\sin\left(\frac{n\pi x}{L}\right)$$

which are the shapes shown earlier.

At a fixed point x,

$$y=\sin\left(\frac{n\pi v}{L}t\right)$$

or a particular x point oscillates up and down with the normal mode frequency.

Fast Image Sequence (like a movie)

```
function z=acoeff(m,L)
if (2*floor(m/2) == m)
  z=0;
 else
  z=(2*L/(m^2*pi^2))*sin(m*pi/2);
 end

function z=aterm(m,L,v,x,t)
z=acoeff(m,L)*sin(m*pi*x/L).*cos(m*pi*v*t/L);

% m-file waveqxt.m
L=1;
v=1;
x=0:0.01:1;
for j = 1:201
  t=(j-1)*0.01;
  sum=0;
  for k=1:100
    sum=sum+aterm(k,L,v,x,t);
  end
  plot(x,sum,'-k');
  axis([-1 2 -.5 .5]);
  pause(0.01)
end
```

Vibrations of a Rectangular Drum

We now choose to look at a rectangular drum because we can handle this boundary easily with a simple extension of our 1-dimensional solutions in cartesian coordinates. We will look at a circular drum later.

The vibration of a 2-dimensional membrane fixed at the boundaries x = 0 , x = a , y = 0 , y = b can be described as follows:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

We choose

$$u(x,y,t)=X(x)Y(y)T(t)$$

SOV substitution gives

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=\frac{1}{c^2}\frac{1}{T}\frac{\partial^2 T}{\partial t^2}$$

The separation constant assignment goes like

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \lambda_x \quad , \quad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = \lambda_y \quad , \quad \frac{1}{c^2}\frac{1}{T}\frac{\partial^2 T}{\partial t^2} = \lambda$$

with

$$\lambda_x + \lambda_y = \lambda \quad , \quad \text{all constants}$$

The solutions are

$$X(x) = A\cos k_x x + B\sin k_x x \quad , \quad k_x^2 = -\lambda_x$$
$$Y(y) = C\cos k_y y + D\sin k_y y \quad , \quad k_y^2 = -\lambda_y$$
$$T(t) = E\cos\alpha t + F\sin\alpha t \quad , \quad \alpha^2 = -\lambda$$

with

$$k_x^2 + k_y^2 = \frac{\alpha^2}{c^2}$$

The boundary conditions give

$$X(0) = 0 \to A = 0 \quad , \quad Y(0) = 0 \to C = 0$$
$$X(a) = 0 \to k_x = \frac{m\pi}{a} = k_m \quad , \quad Y(b) = 0 \to k_y = \frac{n\pi}{b} = k_n$$

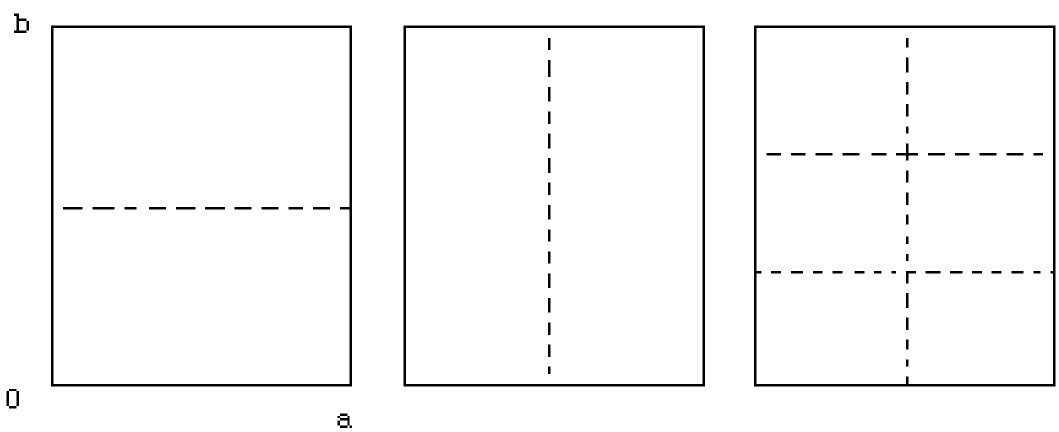
or

$$\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} = \frac{\alpha^2}{c^2} = \frac{\omega_{mn}^2}{c^2} \to \omega_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

The general solution is then a sum of all possible solutions (all m,n)

$$u(x,y,t) = \sum_{m,n=1}^{\infty} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\left(c_{mn}\cos\omega_{mn}t + d_{mn}\sin\omega_{mn}t\right)$$

where ω_{mn} = frequency of the (m,n) normal mode. Some examples of modes are shown below:



(1,2) mode (1,1) mode (2,3) mode

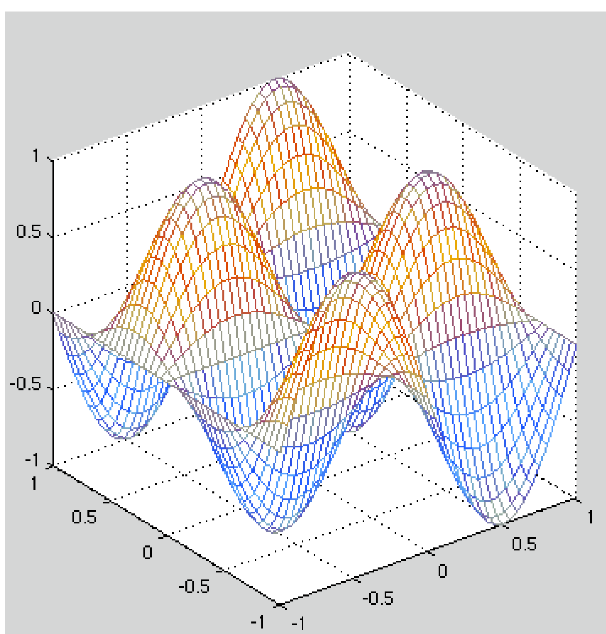
dotted lines = nodal lines all boundaries are nodes

As always, the strength with which various normal modes are excited depends on the exact initial conditions.

The MATLAB program below shows the (n,m) modes of the rectangular membrane (image sequence type movie).

```
x=-1:0.05:1;y=-1:0.05:1;
[X,Y]=meshgrid(x,y);
c=1;a=2;b=2;
% (1,2) mode
;m=1;n=1;
% (1,1) mode
;m=2;n=3;
% (2,3) mode
m=2;n=3;
w=c*pi*sqrt(m^2/a^2+n^2/b^2);
Z=sin(m*pi*X/a).*sin(n*pi*Y/b);
lim=[-1 1 -1 1 -1 1];
figure('Position', [200 200 400 400])
for j=1:500
      t=(j-1)*.01;
      mesh(X,Y,Z*cos(w*t))
      axis(lim);
      colormap(waves);
      pause(0.01);
end
```

An image from the sequence for the (2,3) mode looks like



Diffusion Equation

The 1-dimensional diffusion equation is

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{D}\frac{\partial u(x,t)}{\partial t}$$

Using SOV we have

$$u(x,t) = X(x)T(t)$$
$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{D}\frac{1}{T}\frac{\partial T}{\partial t} = \lambda = \text{separation constant}$$

We get equations and solutions

$$\frac{d^2 X}{dx^2} - \lambda X = 0 \to X(x) = A\cos kx + B\sin kx$$
$$\frac{dT}{dt} - \lambda DT = 0 \to T(t) = Qe^{-k^2 Dt} + \mathrm{Re}^{k^2 Dt}$$

with

$$k^2 = -\lambda$$

The positive exponential solution is not allowed physically since it would imply that, as

$$t \to \infty \Rightarrow T \to \infty$$

which makes no sense in a heat diffusion problem. On the other hand for the negative exponential solution

$$t \to \infty \Rightarrow T \to 0$$

which does make physical sense.

This is just an example of the "physics of the problem" restricting or modifying the strictly mathematical solution.

Therefore, we have

$$T(t) = e^{-k^2 Dt}$$

Special case:

Consider a 1-dimensional rod of length a, at temperature T_0, which has both of its ends placed in contact with a heat reservoir at $T = 0$.

We have the possible solutions

$$u(x,t) = (A\cos kx + B\sin kx)e^{-k^2 Dt}$$

Boundary Conditions

$$x=0 \to u(0,t)=0 \to (A)e^{-k^2 Dt}=0 \to A=0$$

$$x=a \to u(a,t)=0 \to (B\sin ka)e^{-k^2 Dt}=0 \to \sin ka=0 \to k=k_n=\frac{n\pi}{a}$$

Therefore, a solution is given by

$$u_n(x,t)=B_n \sin k_n x\, e^{-k_n^2 Dt}$$

so that the most general solution is

$$u(x,t)=\sum_n B_n \sin k_n x\, e^{-k_n^2 Dt}$$

Initial Conditions

At t = 0, we were in equilibrium such that $T=T_0$ everywhere. This means that

$$\frac{\partial u}{\partial t}=0$$

so that

$$\frac{\partial^2 u(x,0)}{\partial x^2}=\frac{1}{D}\frac{\partial u(x,0)}{\partial t}=0 \to \text{meaning of equilibrium or steady-state}$$

$$\frac{d^2 u(x,0)}{dx^2}=0 \to u(x,0)=F+Gx=T_0 \quad \text{for all } x \to G=0\,,\, F=T_0$$

This gives

$$u(x,0)=T_0=\sum_n B_n \sin k_n x$$

$$B_n=\frac{2}{a}\int_0^a T_0 \sin\frac{n\pi x}{a}dx=\frac{2T_0}{n\pi}(1-\cos n\pi)=\begin{cases} 4T_0/n\pi & \text{n odd} \\ 0 & \text{n even}\end{cases}$$

The final solution is then

$$u(x,t)=T_0\sum_{n\ odd}\frac{4}{n\pi}\sin\frac{n\pi x}{a}e^{-\frac{n^2\pi^2}{a^2}Dt}$$

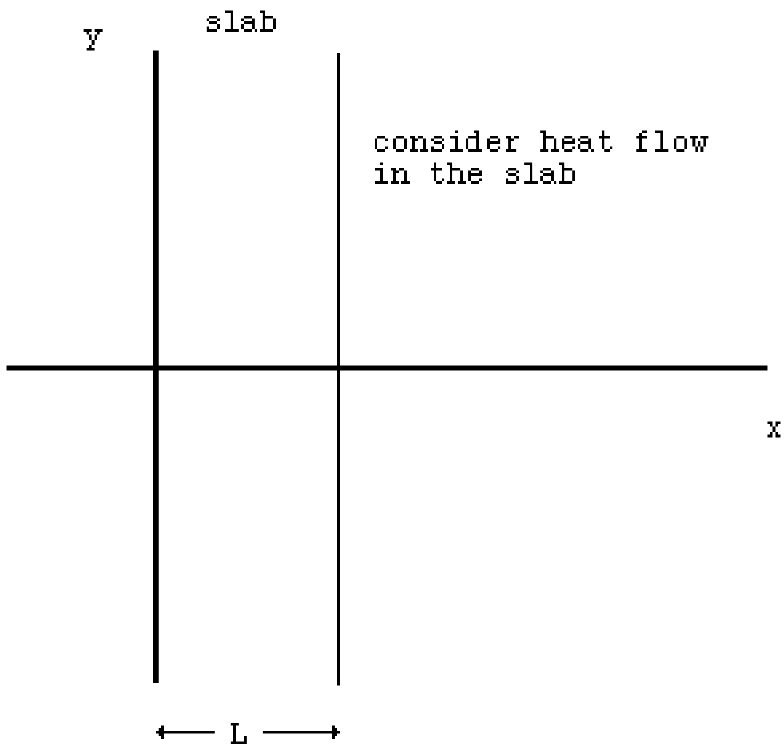
The MATLAB program tempdiff.pro, which generates a sequence of temperature profiles (in time), is given by

```
% m-file diff1.m
D=1.0;
a=1.0;
T0=100.0;
kk=100;
tdel=0.005;
x=(a/kk)*(1:kk);
numterms=500;
for i=1:201
```

```
 t=tdel*(i-1);
 z=0.0;
 for j=1:numterms
  jj=2*j-1;
  z=z+T0*(4.0/(jj*pi))*sin(jj*pi*x/a).*exp(-(jj*jj*pi*pi/a^2)
*D*t);
 end
 plot(x,z,'-k');
 axis([-0.1 1.1*a -10.0 1.1*T0]);
 pause(0.05);
end
```

Another example:

We consider a slab, as shown below, which is infinite in the y-direction. This means we only need to worry about the x and t variables.



We assume that initially (t = 0)

$$T = 0 \quad at \quad x = 0$$

$$T = 100 \quad at \quad x = L$$

and for t > 0 we assume

$$T = 0 \quad at \quad x = 0$$

$$T = 0 \quad at \quad x = L$$

and ask what happens in time?

The $T(t)$ solution is the same as before, namely, $T(t) = e^{-k^2\alpha^2 t}$ where $D = \alpha^2$.

For the x solution, first we find the initial steady-state T distribution. We have only x to worry about (everything must be uniform in y). Now, steady-state means

$$\frac{\partial u}{\partial t} = 0$$

or

$$\frac{d^2u(x,0)}{dx^2} = 0 \rightarrow u(x,0) = ax + b$$
$$u(0,0) = 0 \rightarrow b = 0$$
$$u(L,0) = 100 \rightarrow a = \frac{100}{L}$$

so that

$$u(x,0) = \frac{100x}{L}$$

For t > 0 we use the diffusion equation which gives

$$u(x,t) = (a\cos kx + b\sin kx)e^{-k^2Dt}$$
$$u(0,t) = 0 \rightarrow b = 0$$
$$u(L,t) = 0 \rightarrow \sin kL = 0 \rightarrow k = k_n = \frac{n\pi}{L}$$

Therefore the most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} \sin\frac{n\pi x}{L}$$

Now

$$u(x,0) = \frac{100x}{L} = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}$$

which gives

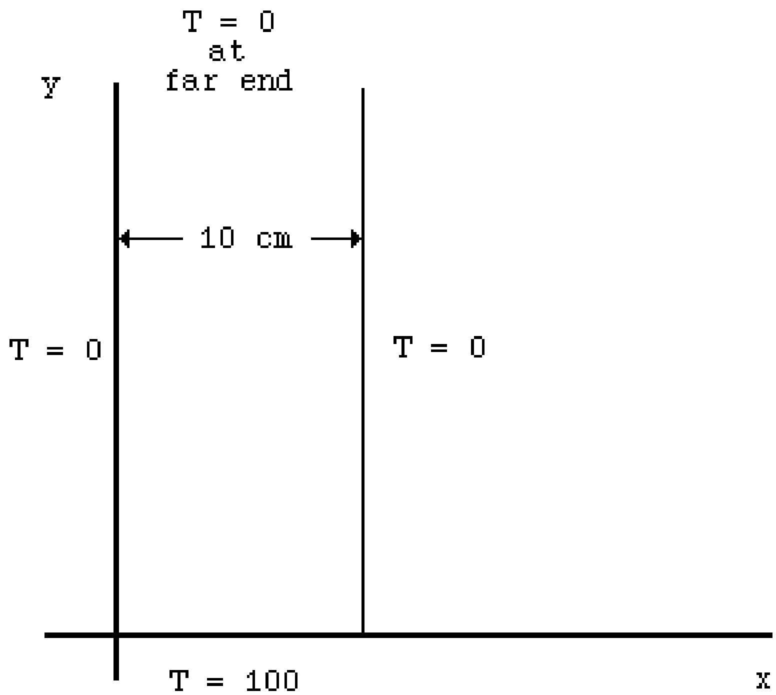
$$a_n = \frac{200}{\pi}\frac{(-1)^{n-1}}{n}$$

and the solution

$$u(x,t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}\sin\frac{n\pi x}{L}$$

Heat conduction example:

We consider a long rectangular metal plate which has the steady-state configuration as shown in the figure below:



We are looking for a solution of the 2-dimensional steady-state diffusion equation

$$\nabla^2 T(x,y) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{D}\frac{\partial T}{\partial t} = 0$$

We assume

$$T(x,y) = X(x)Y(y) \to \frac{d^2 X}{dx^2} = -\frac{d^2 Y}{dy^2} = -k^2 \quad , \quad k \geq 0$$

The solutions are

$$X(x) = a\sin kx + b\cos kx$$
$$Y(y) = ce^{ky} + de^{-ky}$$

We made this particular choice of signs for the separation constant because we need X to be trigonometric functions and Y to be exponential functions for physical reasons, i.e., X being trigonometric is the only way for us to be able to have X = 0 at **both** x = 0 and x = 10.

Therefore, the solution is

$$T(x,y) = (a\sin kx + b\cos k)(ce^{ky} + de^{-ky})$$

The boundary conditions give

$$T(0,y)=0\to b=0$$

$$T(10,y)=0\to \sin 10k=0\to k=k_n=\frac{n\pi}{10}\quad,\quad n=1,2,3,.....$$

which gives the most general solution (sum of all possible solutions) as

$$T(x,y)=\sum_n \sin k_n x(c_n e^{k_n y}+d_n e^{-k_n y})$$

Now, if we let the far end be at y = 30, then we have

$$T(x,30)=0=\sum_n \sin k_n x(c_n e^{30k_n}+d_n e^{-30k_n})\to \frac{c_n}{d_n}=e^{-60k_n}$$

Therefore

$$T(x,y)=\sum_n d_n \sin k_n x(e^{-60k_n}e^{k_n y}+e^{-k_n y})=\sum_n d_n \sin\frac{n\pi x}{10}e^{-3n\pi}(e^{\frac{n\pi}{10}(y-30)}+e^{-\frac{n\pi}{10}(y-30)})$$

$$=\sum_n D_n \sin\frac{n\pi x}{10}\sinh\frac{n\pi}{10}(y-30)$$

Finally, we have

$$T(x,0)=100=-\sum_n D_n \sin\frac{n\pi x}{10}\sinh 3n\pi$$

Solving for D_n we finally get

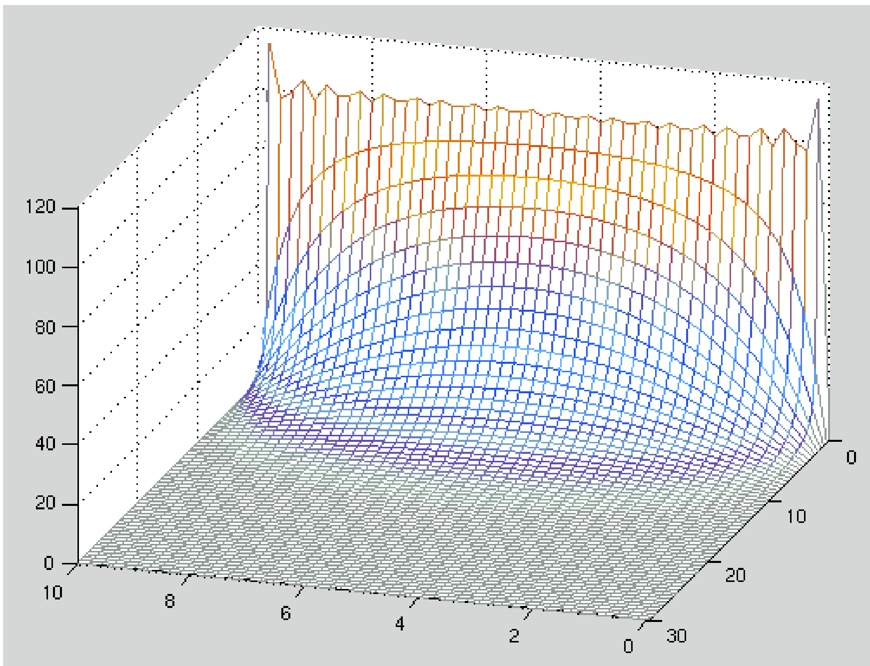
$$T(x,y)=-\sum_{odd\ n}\frac{400}{n\pi\sinh 3n\pi}\sin\frac{n\pi x}{10}\sinh\frac{n\pi}{10}(y-30)$$

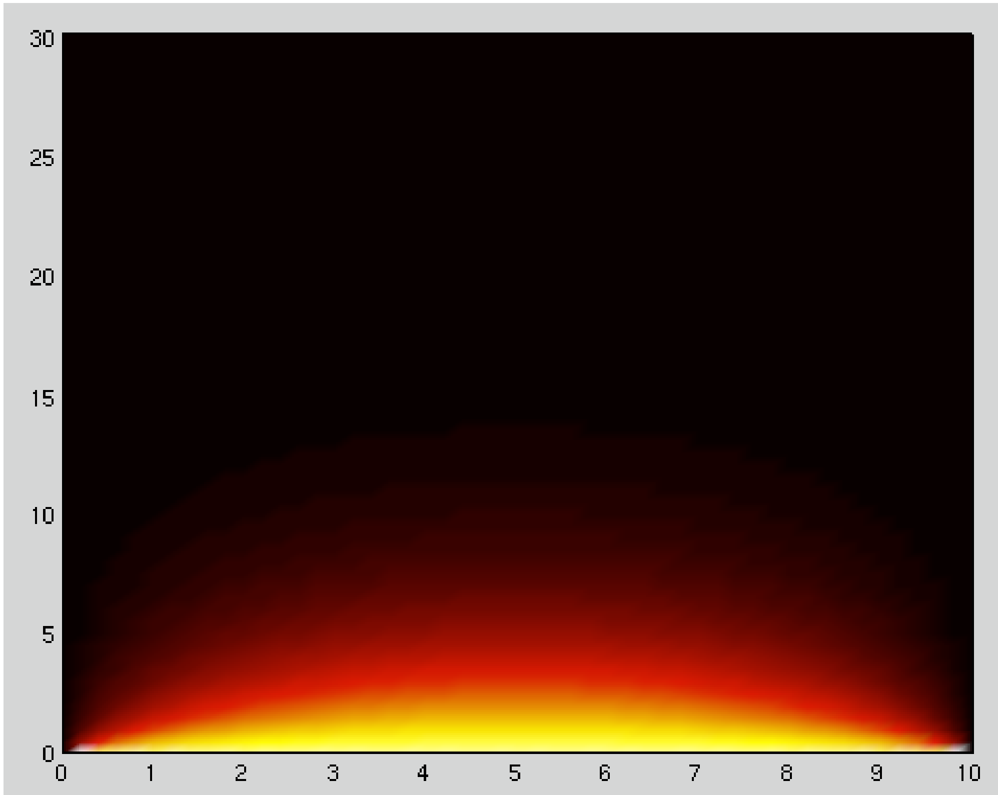
A MATLAB program that plots the steady-state solution as a surface and an image is given by

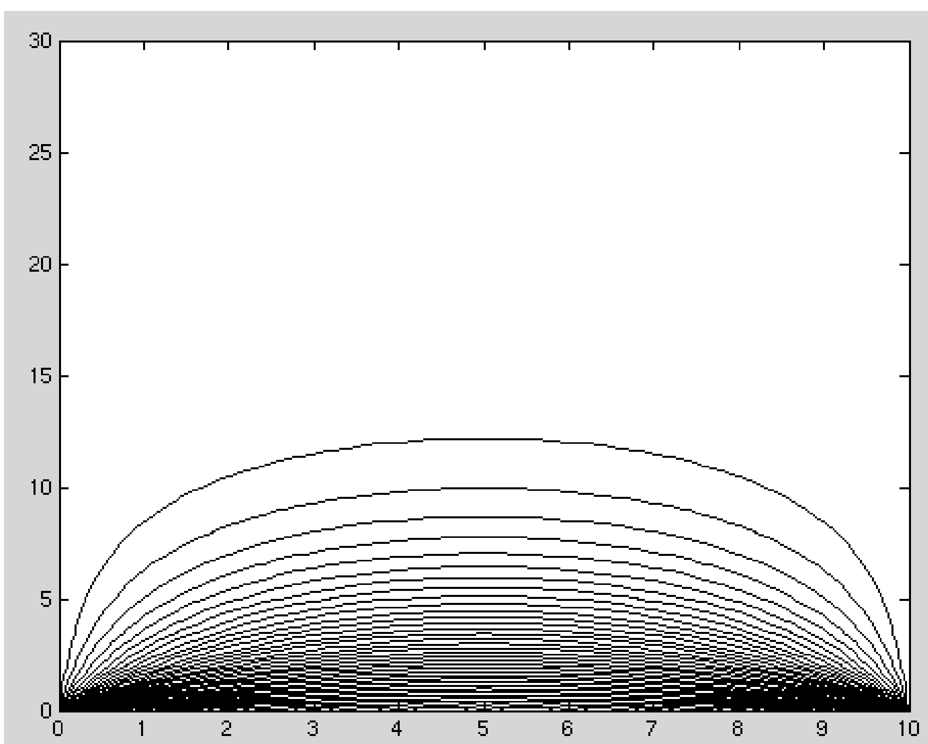
```
function z=ssfunc1(x,y,n)
z=(400.0/(n*pi*sinh(3.0*n*pi)))*sin(n*pi*x/10.0).* ...
           sinh(n*pi*(30.0-y)/10.0);

% m-file diff2.m
x=0.0:0.2:10.0;
y=0.0:0.5:30.0;
[X,Y]=meshgrid(x,y);
sum=zeros(length(y),length(x));
for ii=1:30
  jj=2*ii-1;
  sum=sum+ssfunc1(X,Y,jj);
end
figure
mesh(X,Y,sum);
colormap(waves);
```

```
% use Tools > Rotate3d to orient for viewing
figure
pcolor(X,Y,sum);
colormap(hot);
shading interp
figure
contour(X,Y,sum,40,'-k');
```

The results are




30
25
20
15
10
5
0
0 1 2 3 4 5 6 7 8 9 10


30
25
20
15
10
5
0
0 1 2 3 4 5 6 7 8 9 10