Matrices in Polarization Optics

Polarized Light - Its Production and Analysis

For all electromagnetic radiation, the oscillating components of the electric and magnetic fields are directed at right angles to each other and to the direction of propagation. We assume a right-handed set of axes, with the Oz-axis pointing along the direction of propagation. In dealing with polarization, we can ignore the magnetic field and concentrate on how the electric field vector is oriented with respect to the transverse xy-plane.

Most ordinary sources of light, such as the sun or an incandescent light bulb, produces light that is incoherent and unpolarized; light of this sort is a chaotic jumble of almost innumerable independent disturbances, each with their own direction of travel, its own optical frequency and its own state of polarization. Just how innumerable are these "states" or "modes" of the radiation field?
As far as transverse modes are concerned, for a source of area $A$ radiating into a solid angle $\Omega$, the number of distinguishable directions of travel is given by $A\Omega/\lambda^2$ - for a source of 1 cm$^2$ area radiating into a hemisphere, this will be well over $10^8$. Secondly, if we consider a source which is observed over a period of one second and ask how many optical frequencies, in theory at least, can be distinguished, we find approximately $10^{14}$.

By contrast with these numbers, if we consider the polarization of an electromagnetic wave field, we find the number of independent orthogonal states is just two.  

**Unpolarized** light is sometimes said to be a random mixture of "all sorts" of polarization; it would be better to say that, whenever we try to analyze it, in terms of a chosen orthogonal pair of states, we can find no evidence for preference of one state over the other. If we could make our observations fast enough, we would find that the "instantaneous" state of polarization is passing very rapidly through all possible combinations of the two states we have chosen, in a statistically random way. In most cases, the rate at which these changes occur is over $10^{12}$ per second, so that what we observe is a smoothed average.
Unpolarized light is easy to produce but difficult to describe, whether mathematically or in terms of a model.

At the opposite extreme is the completely coherent light that is generated in a single mode laser. This light is as simple as one could specify. We assume that we have a plane wave of angular frequency \( \omega \) which is traveling with velocity \( c \) in the direction \( Oz \). Since we know that the vibrations of the electric field are transverse, they can be specified in terms of an \( x \)-component \( E_x \), of peak amplitude \( H \), and a \( y \)-component \( E_y \), of peak amplitude \( K \). We thus have

\[
E_x = H \cos\left[ \omega (t - z/c) + \phi_x \right] = \text{Re}\left( H \exp\left( i \left[ \omega (t - z/c) + \phi_x \right] \right) \right)
\]

\[
E_y = V \cos\left[ \omega (t - z/c) + \phi_y \right] = \text{Re}\left( V \exp\left( i \left[ \omega (t - z/c) + \phi_y \right] \right) \right)
\]

We know from our discussions in Physics 8 that these functional forms satisfy the wave equation with wave propagation velocity = \( c \), \( k=2\pi/\lambda=\omega/c \) and wave propagation direction along the \( z \)-axis (Poynting vector direction).

If we use \( \Delta \) to represent the phase difference \( (\phi_y - \phi_x) \), and if the symbols \( \hat{i} \) and \( \hat{j} \) denote unit vectors directed along the axes \( Ox \) and \( Oy \), then these two equations can be combined in the space vector form:

\[
\begin{bmatrix}
E_x \\
E_y
\end{bmatrix} = \begin{bmatrix}
H \\
Ke^{i\Delta}
\end{bmatrix} \exp\left[ i \left[ \omega (t - z/c) + \phi_x \right] \right]
\]

There is no dependence on \( x \) or \( y \) since a plane wave of indefinite lateral extent is assumed.
A column vector or ket such as
\[
\begin{bmatrix}
He^{i\phi_x} \\ Ke^{i\phi_y}
\end{bmatrix}
\] or
\[
\begin{bmatrix}
H \\ Ke^{i\Delta}
\end{bmatrix}
\] is usually referred to as a Jones vector.

The Jones vector provides a complete description of the state of polarization of any light beam that is fully polarized.

For the coherent plane wave described above we see that, if either H or K vanishes, the transverse vibrations must be vertically or horizontally polarized. If the phase difference \(\Delta\) vanishes, the polarization is linear and if \(H = K\), while \(\Delta = \pi/2\) we have light that is circularly polarized. For the general case, the light is elliptically polarized. Some examples are:

\[
H = 1, K = 0, \Delta = 0 \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \text{linear – polarization}(x - 0^\circ)
\]

\[
H = 0, K = 1, \Delta = 0 \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \text{linear – polarization}(y - 90^\circ)
\]

\[
H = \frac{1}{\sqrt{2}}, K = \frac{1}{\sqrt{2}}, \Delta = 0 \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \text{linear – polarization}(45^\circ)
\]

\[
H = \frac{1}{\sqrt{2}}, K = \frac{\pm 1}{\sqrt{2}}, \Delta = 90^\circ \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ \pm i \end{bmatrix} \Rightarrow \begin{bmatrix} \text{right} \\ \text{left} \end{bmatrix} \text{ – circular – polarization}
\]

If these geometrical descriptions of the transverse vibrations are difficult to visualize, it is helpful to think of them in terms of a model. Imagine that, in a given xy-plane, we are following the motion of a very small charged particle, which is simultaneously attached to the origin by a weak spring and pulled away from it by the oscillating electric field vector. If we choose the time-origin so that \(\phi_x = 0\), then the instantaneous x- and y-coordinates of the test charge, its displacement from the origin, will vary with time according to the equations
\[ x = H \cos(\omega t) \quad , \quad y = K \cos(\omega t + \Delta) \]

It follows that, whenever \( \omega t \) changes through a range of \( 2\pi \) (which happens at least \( 10^{14} \) times every second), the test charge executes one cyclic traversal of a single Lissajous figure as shown in Figure 1 below.

Each of these patterns represents the locus of the tip of the oscillating electric field vector. In order to analyze their geometry, we must eliminate the time parameter \( \omega t \) from the above pair of equations.
**Plane-polarized light**

Plane-polarized light is obtained by passing unpolarized light through a polarizing filter. One of the most efficient types of polarizing filters is based on the double-refraction properties of a uniaxial crystal such as calcite.

A **Nicol prism** is a type of polarizer, an optical device used to generate a beam of polarized light. It was the first type of polarizing prism to be invented, in 1828 by William Nicol (1770-1851) of Edinburgh. It consists of a rhombohedral crystal of calcite (Iceland spar) that has been cut at a 68° angle, split diagonally, and then joined again using Canada balsam (see below).

Unpolarized light enters one end of the crystal and is split into two polarized rays by birefringence. One of these rays (the **ordinary** or o-ray) experiences a refractive index of $n_o = 1.658$ and at the balsam layer (refractive index $n = 1.55$) undergoes total internal reflection at the interface, and is reflected to the side of the prism. The other ray (the **extraordinary** or e-ray) experiences a lower refractive index ($n_e = 1.486$), is not reflected at the interface, and leaves through the second half of the prism as plane polarized light.
A **Wollaston prism** is an optical device, invented by William Hyde Wollaston, that manipulates polarized light. It separates randomly polarized or unpolarized light into two orthogonal, linearly polarized outgoing beams. The Wollaston prism consists of two orthogonal calcite prisms, cemented together on their base (typically with Canada balsam) to form two right triangle prisms with perpendicular optic axes. Outgoing light beams diverge from the prism, giving two polarized rays, with the angle of divergence determined by the prisms' wedge angle and the wavelength of the light (see below).

Commercial prisms are available with divergence angles from 15° to about 45°.

Also, remarkably efficient, over a wide range of wavelengths and directions, are the various polarizing sheets of **Polaroid** material.

Imagine that we have converted unpolarized light into plane-polarized light by passing it through a sheet polarizer. The vibrations of the electric field vector now lie entirely in one direction in the transverse xy-plane (the plane of polarization).
Suppose that the direction of polarization makes an angle $\theta$ with the horizontal $x$-axis. The equations describing the transverse electric field components are then of the form:

$$E_x = A \cos \theta \cos(\omega t + \phi), \quad E_y = A \sin \theta \cos(\omega t + \phi)$$

Now consider the hypothetical small test charge once again. Under the influence of this synchronized pair of field vectors, its displacement will be given by

$$x = A \cos \theta \cos(\omega t + \phi), \quad y = A \sin \theta \cos(\omega t + \phi)$$

Multiplying the first equation by $\sin \theta$ and the second by $\cos \theta$ we find that the locus of the transverse displacement is the equation - a straight line motion in the plane of polarization.

**Elliptically Polarized Light**

A convenient method for producing elliptically polarized light is to send a plane-polarized beam through a "phase plate", that is, a slice of uniaxial crystal. Such a slice (as we will see later) introduces a phase difference between the components of the field vectors parallel and perpendicular to a special direction in the crystal, which is known as the **optic(fast) axis**. The field vector parallel to the optic(fast) axis is called the **extraordinary** component (e-ray) and that perpendicular to the optic(fast) axis is called the **ordinary** component (o-ray). For most phase plates, the refractive index for the e-ray is smaller than for the o-ray.

We suppose that the phase plate has its optic(fast) axis parallel to the $x$-axis and that its thickness is such that it advances the o-ray by angle $\Delta$ radians with respect to the e-ray. The field vectors coming out of the phase plate are then given by
\[ x = A \cos \theta \cos(\omega t) , \quad y = A \sin \theta \cos(\omega t + \Delta) \]

If we eliminate \( \omega t \) between these two equations, we get an equation linking the x- and y-components of the field vector in the resultant beam:

\[
\frac{x^2}{A^2 \cos^2 \theta} - \frac{2xy \cos \Delta}{A^2 \cos \theta \sin \theta} + \frac{y^2}{A^2 \sin^2 \theta} = \sin^2 \Delta
\]

or

\[
\frac{x^2}{H^2} - \frac{2xy \cos \Delta}{HK} + \frac{y^2}{K^2} = \sin^2 \Delta
\]

where \( H = A \cos \theta \) and \( K = A \sin \theta \). Remember, \( H \) is the component of the original field vector parallel to the x-axis and \( K \) is the component of the original field vector parallel to the y-axis. Clearly, we have \( H^2 + K^2 = A^2 = I = \text{intensity} \) (proportional to energy flow).

Some special cases are:

If \( \Delta = 0 \) or no phase plate, then \( \cos \Delta = 1 , \sin \Delta = 0 \) and the equation becomes

\[
\frac{x^2}{H^2} - \frac{2xy}{HK} + \frac{y^2}{K^2} = 0 \rightarrow \left( \frac{x}{H} - \frac{y}{K} \right)^2 = 0 \rightarrow \frac{x}{H} = \frac{y}{K}
\]

This just describes the plane-polarized condition of the original beam.

If \( \Delta = \pi \), or a half-wave plate, so that \( \cos \Delta = -1 , \sin \Delta = 0 \), we have

\[
\frac{x}{H} = -\frac{y}{K}
\]
This means that the ratio of the displacement parallel to the two axes is the same as before, but with reversed sign, so that when x is at its maximum positive, y is at its maximum negative, and vice versa. We still have plane-polarized light, but its direction is now an angle $\theta$ on the other side of the optic(fast) axis, that is, the half-wave plate has turned the direction of polarization through an angle of $2\theta$.

If $\Delta = \pi / 2$, or a quarter-wave plate, so that $\cos \Delta = 0$, $\sin \Delta = 1$, the equation linking $x$ and $y$ becomes

$$\frac{x^2}{H^2} + \frac{y^2}{K^2} = 1$$

This is the well-known equation of an ellipse lying with its major and minor axes parallel to the x- and y-axes. The semi-major axis parallel to the x-axis is $H$ and that parallel to the y-axis is $K$. If $\theta = 45^\circ$, then $H$ and $K$ are equal and the equation becomes

$$x^2 + y^2 = \frac{A^2}{2}$$

and we say that the light is circularly polarized.

If a beam of fully polarized light is viewed through a Polaroid sheet (polarizer), and the polarizer is gradually rotated through $360^\circ$, it is quite easy to distinguish experimentally between light that is linearly, elliptically or circularly polarized.

For linear states, there will be two orientations at which the intensity of the light will be zero. For elliptical states, there will be two maxima and two minima of intensity, but the minima will not be zero. For circularly polarized light, the intensity will remain constant.
Any state of polarization which is elliptical or circular is called either "right-handed" or "left-handed". For a right-handed state, the phase angle $\Delta$ by which the y-component leads the x-component must lie between 0 and $\pi$. In that case, the tip of the electric field vector travels around its elliptical path in a clockwise direction as seen from an observer looking from the +z-direction towards the light source (see Figure 1).

**Use of Jones Calculus for Transforming a Ket or Column Vector**

In the Jones calculus, optical devices correspond to 2x2 matrices with complex elements. There are no redundant elements and every Jones matrix that can be written corresponds to a device which, in principle at least, can be physically realized.

The Jones matrices representing devices act on the optical column vectors (kets) described earlier. The elements of these vectors represent the amplitudes and phases of the components of the transverse electric field. The electric field components in the beam leaving the devices are linear functions of the electric field components in the input beam and the matrix which connects the components in the beam leaving the devices with those entering the devices gives a complete (as far as polarization is concerned) characterization of the devices.

We saw earlier the representation of the transverse electric fields corresponding to any fully polarized beam can be represented by a column vector. For a plane wave which is being propagated in the direction +z, the electric field can be regarded as the real part of the complex vector.
In most cases the time-dependence and the z-dependence of the beam are transferred outside the column as a scalar multiplier, so that we have

\[
\begin{bmatrix}
E_x \\
E_y
\end{bmatrix} = \begin{bmatrix}
H \exp\left[ i \left\{ \omega \left( t - \frac{z}{c} \right) + \phi_x \right\} \right] \\
K \exp\left[ i \left\{ \omega \left( t - \frac{z}{c} \right) + \phi_y \right\} \right]
\end{bmatrix}
\]

A useful rule for calculating the intensity of the beam is to premultiply the ket by its hermitian conjugate, an operation known as finding the bra(c)ket product. We obtain

\[
I = \begin{bmatrix}
E_x^* & E_y^*
\end{bmatrix} \begin{bmatrix}
E_x \\
E_y
\end{bmatrix} = \begin{bmatrix}
H \exp[-i\phi_x] & K \exp[-i\phi_y]
\end{bmatrix} \begin{bmatrix}
H \exp[i\phi_x] \\
K \exp[i\phi_y]
\end{bmatrix}
\]

\[
= H^2 + K^2
\]

Often the absolute amplitude is not required and can be transferred outside the column into the scalar multiplier. For a beam of unit intensity, we must have \( H^2 + K^2 = 1 \) so we can write \( H = \cos \theta \) and \( K = \sin \theta \) for some \( \theta \). In terms of this "normalized column vector, the expression for the beam becomes

\[
\begin{bmatrix}
\cos \theta \\
\sin \theta e^{i\Delta}
\end{bmatrix} A \exp\left[ i \left\{ \omega \left( t - \frac{z}{c} \right) \right\} \right]
\]
This expression provides all that is needed to represent a completely coherent plane-monochromatic wave disturbance.

It is an experimental fact that at optical frequencies the response of the medium to an electromagnetic wave field is almost perfectly linear. We can therefore predict that, if

\[
E_1 = \begin{bmatrix} H_1 e^{i\phi_1} \\ K_1 e^{i\psi_1} \end{bmatrix}
\]

represents the ket of the beam entering some kind of polarizing device, the ket representing the beam leaving the device can be represented as

\[
E_2 = \begin{bmatrix} H_2 e^{i\phi_2} \\ K_2 e^{i\psi_2} \end{bmatrix}
\]

and

\[
H_2 e^{i\phi_2} = J_{11} H_1 e^{i\phi_1} + J_{12} K_1 e^{i\psi_1}
\]
\[
K_2 e^{i\psi_2} = J_{21} H_1 e^{i\phi_1} + J_{22} K_1 e^{i\psi_1}
\]

or in matrix form

\[
\begin{bmatrix} H_2 e^{i\phi_2} \\ K_2 e^{i\psi_2} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} H_1 e^{i\phi_1} \\ K_1 e^{i\psi_1} \end{bmatrix}
\]

In general, the four elements of the 2x2 matrix are complex and depend only on the device.
It will be noticed that in the equations above, we have, for the sake of generality, used two separate phase angles $\Phi$ and $\Psi$ in each of the kets. As far as the state of polarization is concerned, it is only the phase difference $(\Phi-\Psi)=\Delta$ which has any physical significance. But it is not always possible to ensure, even for an input for which the phase angle $\phi_1$ is zero, that the corresponding output phase angle $\phi_2$ for the x-component is also zero. Once and output ket has been calculated, it is very easy to multiply the whole ket by any desired phase factor.

Starting with the above equation, we can now develop the Jones calculus. The matrix consisting of the four $J_{ij}$'s is called the **Jones matrix** $J$ of the device, so that the matrix equation can be written $E_2 = JE_1$.

Suppose now we have two devices, whose Jones matrices are $J_a$ and $J_b$. We pass a beam of light through the two devices in series. Let the ket of the original beam of light be $E_1$, that of the beam between the devices $E_2$ and that after the second device $E_3$. Then, we have $E_2 = J_a E_1$ and $E_3 = J_b E_2$. Substituting the first of these equations into the second, we get

$$E_3 = J_b (J_a E_1) = (J_b J_a) E_1$$

using the associativity property of matrix multiplication. Thus, we see that the effect of a number of devices in series on a beam of light can be obtained by multiplying their Jones matrices together.

**Derivation of Jones Matrices** - For simplicity we will work with equation of the form
\[
H_2 = J_{11}H_1 + J_{12}K_1e^{i\Delta_1}
\]

\[
K_2 = J_{21}H_1 + J_{22}K_1e^{i\Delta_1}
\]

which is sufficient for all examples we will consider. We note the connection using the form

\[
\begin{bmatrix}
\cos \theta \\
\sin \theta e^{i\Delta}
\end{bmatrix}
= \begin{bmatrix}
H \\
K e^{i\Delta}
\end{bmatrix}
\]

The Polarizer (such as a sheet of Polaroid)

(a) First, we consider an ideal linear polarizer, whose pass-plane is horizontal, that is, parallel to the x-direction. This allows only electric field vectors parallel to the x-direction to pass through the device, so that in the general equations describing the behavior of an ideal device \(H_2\) is equal to \(H_1\) and \(K_2\) is zero, no matter what the values of \(H_1\) and \(K_1\) may be. Thus, the equations become

\[
H_2 = H_1 = J_{11}H_1 + J_{12}K_1e^{i\Delta_1}
\]

\[
K_2 = 0 = J_{21}H_1 + J_{22}K_1e^{i\Delta_1}
\]

for all values of \(H_1\), \(K_1\), and \(\Delta_1\).

Set \(K_1=0\). The equations become \(H_1=J_{11}H_1\) or \(J_{11}=1\) and \(0=J_{21}H_1\) for all \(H_1\), that is, \(J_{21} = 0\).
Set $H_1=0$. The equations become $0=J_{12}K_1\exp(i\Delta_1)$ and $0=J_{22}K_1\exp(i\Delta_1)$ for all $K_1$ and $\Delta_1$, that is, $J_{12}$ and $J_{22}$ are both zero. The Jones matrix for this ideal x-polarizer is thus

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = P_x = P_{\theta=0}$$

In practice, of course, a sheet of Polaroid introduces some attenuation, even for the preferred direction of polarization and its optical thickness will be sufficient to introduce at least several hundred wavelengths of retardation. Thus, for some purposes, for example, in an interferometer calculation, the above matrix needs to be multiplied by a complex scalar, which represents the complex amplitude transmittance of the polarizer.

The definition of the Jones matrix is arbitrary up to an over all phase factor(intensity only). Thus, a general representation of the Jones matrix for this ideal polarizer is

$$\begin{bmatrix} e^{i\delta} & 0 \\ 0 & 0 \end{bmatrix} = P_x = P_{\theta=0}$$

In a similar manner we can show that, for a polarizer with its pass-plane vertical, that is, parallel to the y-axis, the Jones matrix can be represented by

$$\begin{bmatrix} 0 & 0 \\ 0 & e^{i\alpha} \end{bmatrix} = P_y = P_{\theta=\pi/2}$$
(b) We now consider the more general case of a polarizer with its pass-plane making an angle $\theta$ with the x-direction as shown in Figure 2 below.

Suppose we have a plane-polarized electric field incident such that its polarization direction makes an angle $\alpha$ with the x-axis. Suppose the amplitude of this incident beam is $A$. Then, by definition $X_1 = A \cos \alpha$ and $Y_1 = A \sin \alpha$. Only the component of the electric field along the direction of the pass-plane of the Polaroid will pass through the device. The amplitude of this component is

$$U = A \cos(\alpha - \theta) = A \cos \alpha \cos \theta + A \sin \alpha \sin \theta$$

$$= X_1 \cos \theta + Y_1 \sin \theta$$

The component of the emerging vector parallel to the x-axis is

$$X_2 = U \cos \theta = X_1 \cos^2 \theta + Y_1 \sin \theta \cos \theta$$

The component of the emerging vector parallel to the y-axis is

$$Y_2 = U \sin \theta = X_1 \sin \theta \cos \theta + Y_1 \sin^2 \theta$$
Writing the last two equation in matrix form we have

$$\begin{bmatrix} X_2 \\ Y_2 \end{bmatrix} = \begin{bmatrix} \cos^2 \theta & \sin \theta \cos \theta \\ \sin \theta \cos \theta & \sin^2 \theta \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix}$$

so the matrix for this device is

$$\begin{bmatrix} \cos^2 \theta & \sin \theta \cos \theta \\ \sin \theta \cos \theta & \sin^2 \theta \end{bmatrix} = P_\theta$$

**Jones Matrix of the General Linear Retarder**

We now consider a crystal plate with its optic(fast) axis at an angle $\alpha$ to the x-axis and we suppose that this plate retards the phase of the o-ray (orthogonal to the optic(fast) axis) by $\delta$ relative to the field component along the optic(fast) axis, that is, to the e-ray. This device makes no difference to the ket of a plane-polarized vibration parallel to the optic(fast) axis. Suppose the amplitude of this field is $A$, then, as in the discussion of the Polaroid, we know that the components are $X_1 = A \cos \alpha$ and $Y_1 = A \sin \alpha$. Since this field component is unaffected by the device, the emerging components are the same as the entering ones, that is, $X_2 = A \cos \alpha$ and $Y_2 = A \sin \alpha$. Substituting in the general equation for the behavior of the device, we find the equations

$$A \cos \alpha = J_{11} A \cos \alpha + J_{12} A \sin \alpha$$
$$A \sin \alpha = J_{21} A \cos \alpha + J_{22} A \sin \alpha$$

which give
as long as $\cos \alpha$ does not vanish. We now consider the action of the device on a beam of plane-polarized light with its direction of polarization parallel to the x-axis and of amplitude $A$.

**Digression to rotation matrices:**

**Vector Rotation**

Consider a vector

$$\vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 = A \cos \theta \hat{e}_1 + A \sin \theta \hat{e}_2$$

Now rotate the vector through an angle $\Phi$ as shown:

We then have

$$\vec{A}' = A'_1 \hat{e}_1 + A'_2 \hat{e}_2 = A \cos(\theta + \phi) \hat{e}_1 + A \sin(\theta + \phi) \hat{e}_2$$

Therefore we find that the components transform as

$$A'_1 = A_1 \cos \phi - A_2 \sin \phi$$

$$A'_2 = A_1 \sin \phi + A_2 \cos \phi$$
Using matrix notation we have
\[
\begin{bmatrix}
A'
\end{bmatrix} = \begin{bmatrix} R(\phi) \end{bmatrix} \begin{bmatrix} A \end{bmatrix}
\]
\[
\begin{bmatrix}
A'_1 \\
A'_2
\end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{bmatrix} \begin{bmatrix}
A_1 \\
A_2
\end{bmatrix}
\]

The matrix \( \begin{bmatrix} R(\phi) \end{bmatrix} \) is called the rotation matrix and is defined by (for a rotation about the z-axis or 3-axis) as
\[
\begin{bmatrix} R(\phi) \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{bmatrix}
\]

In summation notation we have
\[
A'_i = R_{ij}A_j \rightarrow \vec{A}' = R_{ij}A_j \hat{e}_i
\]

The rotation matrix is an orthogonal matrix since
\[
\begin{bmatrix} R(\phi) \end{bmatrix}^+ \begin{bmatrix} R(\phi) \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi \\
-\sin \phi & \cos \phi
\end{bmatrix} \begin{bmatrix} \cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{bmatrix} = \begin{bmatrix} 1 & 0 \\
0 & 1
\end{bmatrix} = \begin{bmatrix} I \end{bmatrix} = \text{identity matrix}
\]

This says that the length of the vector does not change under the rotation transformation, i.e.,
\[
\vec{A} \cdot \vec{A}' = (R_{ij} A_j \hat{e}_i) \cdot (R_{mn} A_n \hat{e}_m) = A_j A_n R_{ij} R_{mn} \hat{e}_i \cdot \hat{e}_m = A_j A_n R_{ij} R_{mn} \delta_{im}
\]

\[
= A_j A_n R_{ij} R_{in} = A_j A_n ( [R^+]_{ji} R_{in} = A_j A_n \left( [R^+ R] \right)_{jn} = A_j A_n \delta_{jn} = A_j A_j
\]

\[
= \vec{A} \cdot \vec{A}
\]

Continuing: For this plane-polarized beam, the ket is

\[
E_1 = \begin{bmatrix} A \\ 0 \end{bmatrix}
\]

We now need to consider the components of this field orthogonal and parallel to the optic(fast) axis.

To find these components we rotate the axes:

\[
\begin{bmatrix}
\text{component along optic axis} \\
\text{component perpendicular to optic axis}
\end{bmatrix} = \begin{bmatrix} U_1 \\ V_1 \end{bmatrix} = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} A \\ 0 \end{bmatrix} = \begin{bmatrix} A \cos \alpha \\ -A \sin \alpha \end{bmatrix}
\]

The component perpendicular to the optic(fast) axis is now retarded by a phase angle \( \delta \), that is to say \( V_1 \) is multiplied by \( e^{-i\delta} \). The matrix of components parallel and orthogonal to the optic(fast) axis thus become
\[
\begin{bmatrix}
U_2 \\
V_2
\end{bmatrix} =
\begin{bmatrix}
A \cos \alpha \\
-A e^{-i\delta} \sin \alpha
\end{bmatrix}
\]

We now need to return to the original axes. We thus rotate from the UV axes back to the XY axes. The final form of the ket is thus

\[
E_2 = \begin{bmatrix}
\cos \alpha & -\sin \alpha \\
\sin \alpha & \cos \alpha
\end{bmatrix}
\begin{bmatrix}
A \cos \alpha \\
-A e^{-i\delta} \sin \alpha
\end{bmatrix}
\]

\[
= \begin{bmatrix}
A \cos^2 \alpha + A e^{-i\delta} \sin^2 \alpha \\
A \cos \alpha \sin \alpha - A e^{-i\delta} \cos \alpha \sin \alpha
\end{bmatrix}
\]

\[
= \begin{bmatrix}
A(\cos^2 \alpha + e^{-i\delta} \sin^2 \alpha) \\
A \cos \alpha \sin \alpha(1 - e^{-i\delta})
\end{bmatrix}
\]

Now we must have

\[
E_2 = \begin{bmatrix}
J_{11} & J_{12} \\
J_{21} & J_{22}
\end{bmatrix}
\begin{bmatrix}
A \\
0
\end{bmatrix} = \begin{bmatrix}
J_{11} A \\
J_{21} A
\end{bmatrix}
\]

Thus,

\[
J_{11} = \cos^2 \alpha + e^{-i\delta} \sin^2 \alpha
\]

\[
J_{21} = \cos \alpha \sin \alpha(1 - e^{-i\delta})
\]
Using the two equations we derived earlier for $\tan \alpha$

$$\tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \frac{1 - J_{11}}{J_{12}} = \frac{J_{21}}{1 - J_{22}}$$

we find

$$J_{22} = \sin^2 \alpha + e^{-i\delta} \cos^2 \alpha$$

$$J_{12} = \cos \alpha \sin \alpha (1 - e^{-i\delta})$$

so that the Jones matrix of the general phase plate in general orientation $\alpha$ is

$$\begin{bmatrix}
\cos^2 \alpha + e^{-i\delta} \sin^2 \alpha & \cos \alpha \sin \alpha (1 - e^{-i\delta}) \\
\cos \alpha \sin \alpha (1 - e^{-i\delta}) & \sin^2 \alpha + e^{-i\delta} \cos^2 \alpha
\end{bmatrix}$$

The Jones matrix for a simple rotator, that is, a device which twists the polarization direction of a beam of plane-polarized light through angle $\theta$ counterclockwise (turpentine) is

$$R(-\theta) = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}$$

In general, if $J$ represents the Jones matrix already calculated for a particular device, the new matrix for the same device rotated through an angle $\theta$ will be given by the triple matrix product

$$R(-\theta)JR(\theta)$$

As an example of this rule, one can verify that the general retarder at angle $\theta$ is equal to the product.
\[
\begin{bmatrix}
\cos\theta & -\sin\theta \\
\sin\theta & \cos\theta
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & e^{-i\delta}
\end{bmatrix}
\begin{bmatrix}
\cos\theta & \sin\theta \\
-\sin\theta & \cos\theta
\end{bmatrix}
\]

which agree with our earlier result.

The table below gives the Jones matrices for ideal linear polarizers, linear retarders, rotation of axes and circular retarders. The angle $\theta$ specifies how the pass-plane of a polarizer or the optic(fast) axis of a linear retarder is oriented with respect to the x-axis.

<table>
<thead>
<tr>
<th>Type of device</th>
<th>$\theta = 0$</th>
<th>$\theta = \pm\pi / 4$</th>
<th>$\theta = \pi / 2$</th>
<th>$\theta = \text{any value}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ideal linear polarizer at angle $\theta$</td>
<td>$\begin{bmatrix} 1 &amp; 0 \ 0 &amp; 0 \end{bmatrix}$</td>
<td>$\frac{1}{2} \begin{bmatrix} 1 &amp; \pm 1 \ \pm 1 &amp; 1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 &amp; 0 \ 0 &amp; 1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} C_1^2 &amp; C_1S_1 \ C_1S_1 &amp; S_1^2 \end{bmatrix}$</td>
</tr>
<tr>
<td>Quarter-wave linear retarder with optic axis at angle $\theta$</td>
<td>$\begin{bmatrix} 1 &amp; 0 \ 0 &amp; -i \end{bmatrix}$</td>
<td>$\frac{1}{2} \begin{bmatrix} 1-i &amp; \pm(1+i) \ \pm(1+i) &amp; 1-i \end{bmatrix}$</td>
<td>$\begin{bmatrix} -i &amp; 0 \ 0 &amp; 1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} C_1^2 - iS_1^2 &amp; C_1S_1(1+i) \ C_1S_1(1+i) &amp; -iC_1^2 + S_1^2 \end{bmatrix}$</td>
</tr>
</tbody>
</table>

$C_1 = \cos \theta$ , $S_1 = \sin \theta$

The matrix for $\theta = \pm\pi / 4$ can be multiplied by $e^{i\pi/4}$ to give

\[
\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \pm i \\ \pm i & 1 \end{bmatrix}
\]
<table>
<thead>
<tr>
<th>Type of device</th>
<th>$\theta = 0$</th>
<th>$\theta = \pm \pi / 4$</th>
<th>$\theta = \pi / 2$</th>
<th>$\theta = \text{any value}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Half-wave linear retarder with optic axis at angle $\theta$</td>
<td>$\begin{bmatrix} 1 &amp; 0 \ 0 &amp; -1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 &amp; \pm 1 \ \pm 1 &amp; 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} -1 &amp; 0 \ 0 &amp; 1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} C_2 &amp; S_2 \ S_2 &amp; -C_2 \end{bmatrix}$</td>
</tr>
<tr>
<td>Linear retarder with retardation $\delta$ and with optic axis at angle $\theta$</td>
<td>$\begin{bmatrix} 1 &amp; 0 \ 0 &amp; e^{-i\delta} \end{bmatrix}$</td>
<td>$\frac{1}{2} \begin{bmatrix} e^{-i\delta} + 1 &amp; \pm \left(1 - e^{-i\delta}\right) \ \pm \left(1 - e^{-i\delta}\right) &amp; e^{-i\delta} + 1 \end{bmatrix}$</td>
<td>$e^{-i\delta} \begin{bmatrix} 0 \ 1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} e^{i\delta} &amp; 0 \ 0 &amp; 1 \end{bmatrix}$</td>
</tr>
</tbody>
</table>

or

$e^{-i\delta/2} \begin{bmatrix} \cos \frac{\delta}{2} & \pm i \sin \frac{\delta}{2} \\ \pm i \sin \frac{\delta}{2} & \cos \frac{\delta}{2} \end{bmatrix}$

$C_2 = \cos 2\theta$, $S_2 = \sin 2\theta$

**Experimental Determination of Elements of Jones Matrix and Light Column Vector**

We now describe methods for determining the ket vector of any beam of polarized light, whether plane or elliptically polarized, and method by which the Jones matrix of any device can be found experimentally.

These methods involve passing beams of light through the device and through various Polaroids and phase plates, and then measuring the intensity of each beam that emerges.
First, we consider determination of the ket vector of a beam of light. Suppose the ket vector of the beam is

\[
\begin{bmatrix}
H \\
Ke^{i\Delta}
\end{bmatrix}
\]

so that the intensity is

\[
\begin{bmatrix}
H & Ke^{-i\Delta}
\end{bmatrix}
\begin{bmatrix}
H \\
Ke^{i\Delta}
\end{bmatrix} = H^2 + K^2 = 1
\]

that is, we assume unit intensity.

First, we pass the beam through a polaroid with its pass-plane horizontal, that is, parallel to the x-axis. The ket vector of the emerging beam is

\[
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
H \\
Ke^{i\Delta}
\end{bmatrix} = \begin{bmatrix}
H \\
0
\end{bmatrix}
\]

Therefore, the intensity \( I_1 = H^2 \), so that \( H = \sqrt{I_1} \), where \( I_1 \) is the measured intensity.

We now pass the original beam through a polaroid with its pass-plane vertical (parallel to the y-axis), so that the ket vector becomes

\[
\begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
H \\
Ke^{i\Delta}
\end{bmatrix} = \begin{bmatrix}
0 \\
Ke^{i\Delta}
\end{bmatrix}
\]

Therefore, the intensity \( I_2 = K^2 \), so that \( K = \sqrt{I_2} \), where \( I_2 \) is the measured intensity.
We now pass the beam through a polaroid with its pass-plane at an angle $45^\circ$ with the x-axis in the first and third quadrants for which the Jones matrix is

\[
\begin{bmatrix}
\cos^2 45 & \cos 45 \sin 45 \\
\cos 45 \sin 45 & \sin^2 45
\end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
\]

The new ket vector is

\[
\frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} H \\ Ke^{i\Delta} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} H + Ke^{i\Delta} \\ H + Ke^{i\Delta} \end{bmatrix}
\]

The intensity is

\[
I_3 = \frac{1}{2} \left( H^2 + 2HK \cos \Delta + K^2 \right)
\]

We now pass the beam through a polaroid with its pass-plane at angle $45^\circ$ with the x-axis in the second and fourth quadrants for which the Jones matrix is

\[
\begin{bmatrix}
\cos^2(-45) & \cos(-45) \sin(-45) \\
\cos(-45) \sin(-45) & \sin^2(-45)
\end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
\]

The emerging ket vector is

\[
\frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} H \\ Ke^{i\Delta} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} H - Ke^{i\Delta} \\ -H + Ke^{i\Delta} \end{bmatrix}
\]

The intensity is

\[
I_4 = \frac{1}{2} \left( H^2 - 2HK \cos \Delta + K^2 \right)
\]
Therefore

\[ I_3 - I_4 = 2HK \cos \Delta \]

The values of H and K are already known, so this equation determines \( \cos \Delta \), but \( \Delta \) can still be positive or negative. There is an uncertainty of sign, so we now determine \( \sin \Delta \) to resolve this uncertainty.

We pass the original beam through a quarter-wave plate whose optic (fast) axis is horizontal. Using the fact that \( \delta = 90^\circ \) for a quarter-wave plate and that \( \theta = 0^\circ \), we find that the Jones matrix is

\[
\begin{bmatrix}
1 & 0 \\
0 & -i
\end{bmatrix}
\]

and the ket vector of the beam becomes

\[
\begin{bmatrix}
H \\
Ke^{i\Delta}
\end{bmatrix}
\]

We put the beam from the quarter-wave plate through a polaroid inclined at 45° to the axis. The ket vector becomes

\[
\frac{1}{2}
\begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
H \\
-Ke^{i\Delta}
\end{bmatrix} = \frac{1}{2}
\begin{bmatrix}
H - iKe^{i\Delta} \\
H - iKe^{i\Delta}
\end{bmatrix}
\]

The intensity is

\[ I_5 = \frac{1}{2} \left( H^2 - 2HK \sin \Delta + K^2 \right) \]
We now pass the beam from the quarter-wave plate through a polaroid with its pass-plane at angle 45° with the x-axis in the second and fourth quadrants for which the Jones matrix is

\[
\begin{bmatrix}
\cos^2(-45) & \cos(-45)\sin(-45) \\
\cos(-45)\sin(-45) & \sin^2(-45)
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}
\]

The emerging ket vector is

\[
\frac{1}{2} \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix} \begin{bmatrix}
H \\
-\text{i}Ke^{i\Delta}
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
H + Ke^{i\Delta} \\
-H - Ke^{i\Delta}
\end{bmatrix}
\]

The intensity is

\[
I_6 = \frac{1}{2} \left( H^2 + 2HK \sin \Delta + K^2 \right)
\]

Therefore,

\[
I_6 - I_5 = 2HK \sin \Delta
\]

This determines \( \sin \Delta \). Knowing \( \sin \Delta \) and \( \cos \Delta \), we obtain the value of \( \Delta \). We now know all variables of the ket vector of the original beam.

We now describe the method by which the Jones matrix of any device can be determined from intensity measurements. Suppose the Jones matrix of the device is
\[
J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} X_{11} + iY_{11} & X_{12} + iY_{12} \\ X_{21} + iY_{21} & X_{22} + iY_{22} \end{bmatrix} = \begin{bmatrix} R_{11}e^{i\theta_{11}} & R_{12}e^{i\theta_{12}} \\ R_{21}e^{i\theta_{21}} & R_{22}e^{i\theta_{22}} \end{bmatrix}
\]

A. We pass into the device a beam of unit intensity plane-polarized parallel to the x-axis so the incident ket vector is
\[
\begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

On emerging from the device the ket vector will be
\[
\begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} J_{11} \\ J_{21} \end{bmatrix}
\]

A.1. The beam is now passed through a polaroid with its pass-plane horizontal and the ket vector becomes
\[
\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} J_{11} \\ J_{21} \end{bmatrix} = \begin{bmatrix} J_{11} \\ 0 \end{bmatrix}
\]

The intensity is
\[
I_2 = J_{11}^*J_{11} = X_{11}^2 + Y_{11}^2 = R_{11}^2
\]

A.2. The beam from the device is now put through a polaroid with its pass-plane vertical. The ket vector becomes
\[
\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} J_{11} \\ J_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ J_{21} \end{bmatrix}
\]
The intensity is

\[ I_3 = R_{21}^2 \]

B. A beam of unit intensity polarized parallel to the y-axis and having a ket vector

\[
\begin{bmatrix}
0 \\
1
\end{bmatrix}
\]

is now passed into the device. The emerging ket vector is

\[
\begin{bmatrix}
J_{11} & J_{12} \\
J_{21} & J_{22}
\end{bmatrix}\begin{bmatrix}
0 \\
1
\end{bmatrix} = \begin{bmatrix}
J_{12} \\
J_{22}
\end{bmatrix}
\]

B.1. The beam emerging from the device is now passed through a polaroid with its pass-plane horizontal. The emerging ket vector is

\[
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}\begin{bmatrix}
J_{12} \\
J_{22}
\end{bmatrix} = \begin{bmatrix}
J_{12} \\
0
\end{bmatrix}
\]

The intensity is

\[ I_4 = R_{12}^2 \]

B.2. The beam emerging from the device is now passed through a polaroid with its pass-plane vertical. The emerging ket vector is

\[
\begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}\begin{bmatrix}
J_{12} \\
J_{22}
\end{bmatrix} = \begin{bmatrix}
0 \\
J_{22}
\end{bmatrix}
\]
The intensity is 
\[ I_5 = R_{22}^2 \]

We have now determined the magnitude of all four Jones matrix elements. It remains only to find the angles (phase) in these matrix elements.

C. We now pass into the device a beam of right-handed circularly polarized light of unit intensity so that
\[ H = K, \quad H^2 + K^2 = 1 \]

Thus,
\[ H = K = \frac{1}{\sqrt{2}} \]

The ket vector of the incident beam is therefore
\[ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} \]

On emerging from the device the ket vector of the beam is
\[ \frac{1}{\sqrt{2}} \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} J_{11} + iJ_{12} \\ J_{21} + iJ_{22} \end{bmatrix} \]

C.1. The beam is now passed through a polaroid with its pass-plane horizontal so the ket vector becomes
\[ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} J_{11} + iJ_{12} \\ J_{21} + iJ_{22} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} J_{11} + iJ_{12} \\ 0 \end{bmatrix} \]
\[ = \left[ (X_{11} - Y_{12}) + i(Y_{11} + X_{12}) \right] \]
The intensity is thus,

\[ I_6 = \frac{1}{2} \left[ (X_{11} - Y_{12})^2 + (Y_{11} + X_{12})^2 \right] \]

Thus,

\[ \frac{2I_6 - I_2 - I_4}{\sqrt{I_2I_4}} = 2 \sin(\theta_{11} - \theta_{12}) \]

C.2. The beam from the device is now passed through a polaroid with its pass-plane vertical. The ket vector becomes

\[
\frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} J_{11} + iJ_{12} \\ J_{21} + iJ_{22} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ J_{21} + iJ_{22} \end{bmatrix}
\]

so that the intensity is

\[ I_7 = \frac{1}{2} \left[ (X_{21} - Y_{22})^2 + (Y_{21} + X_{22})^2 \right] \]

and thus

\[ \frac{2I_7 - I_3 - I_5}{\sqrt{I_3I_5}} = 2 \sin(\theta_{21} - \theta_{22}) \]

We now know the sines of the angles \((\theta_{11} - \theta_{12})\) and \((\theta_{21} - \theta_{22})\), but this still leaves doubt as to the value of the angles because \(\sin(\pi - \theta) = \sin \theta\). To determine the angles completely we need to know the cosines of the angles as well.

D. We pass into the device a beam of light of unit intensity plane polarized at 45° to the x-axis. For this beam
\[ H = K = \frac{1}{\sqrt{2}} \]

and \( \Delta = 0 \), so that \( e^{i\Delta} = 1 \). For the original beam, the ket vector is

\[
\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

so that after the device the ket vector is

\[
\frac{1}{\sqrt{2}} \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} J_{11} + J_{12} \\ J_{21} + J_{22} \end{bmatrix}
\]

D.1. The beam is now passed through a polaroid with its pass-plane horizontal so the ket vector becomes

\[
\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} J_{11} + J_{12} \\ J_{21} + J_{22} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} J_{11} + J_{12} \\ 0 \end{bmatrix}
\]

and the intensity is

\[
I_9 = \frac{1}{2} \left[ \left( X_{21} + X_{22} \right)^2 + \left( Y_{21} + X_{22} \right)^2 \right]
\]

and thus

\[
\frac{2I_9 - I_3 - I_5}{\sqrt{I_3 I_5}} = 2 \cos(\theta_{22} - \theta_{21})
\]
We now completely know the angles \((\theta_{11} - \theta_{12})\) and \((\theta_{21} - \theta_{22})\).

To determine the individual angles we put a beam that is plane-polarized horizontally into the device so that as earlier the ket emerging from the device is

\[
\begin{bmatrix}
J_{11} \\
J_{21}
\end{bmatrix}
\]

We pass this beam through a polaroid with its pass-plane inclined at 45° to the axis. The ket vector becomes

\[
\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} J_{11} \\
J_{21}\end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} J_{11} + J_{21} \\
J_{11} + J_{21}\end{bmatrix}
\]

We now put the beam through another polaroid with its axis horizontal. The ket vector becomes

\[
\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} J_{11} + J_{12} \\
J_{21} + J_{22}\end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} J_{11} + J_{12} \\
0\end{bmatrix}
\]

As in D.1 or D.2, the intensity becomes

\[
I_{10} = \frac{1}{4} \left[ (X_{11} + X_{21})^2 + (Y_{11} + X_{21})^2 \right]
\]

so that

\[
\frac{4I_{10} - I_2 - I_3}{\sqrt{I_2 I_3}} = 2 \cos(\theta_{11} - \theta_{21})
\]
We now put the same beam from the device
\[
\begin{bmatrix}
J_{11} \\
J_{21}
\end{bmatrix}
\]
through a quarter-wave plate with its optic (fast) axis vertical. The ket vector becomes
\[
\begin{bmatrix}
1 & 0 \\
0 & i
\end{bmatrix}
\begin{bmatrix}
J_{11} \\
J_{21}
\end{bmatrix}
= \begin{bmatrix}
J_{11} \\
iJ_{21}
\end{bmatrix}
\]

We now pass the beam through a polaroid with its axis at 45°. The ket vector becomes
\[
\frac{1}{2}
\begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
J_{11} \\
iJ_{21}
\end{bmatrix}
= \frac{1}{2}
\begin{bmatrix}
J_{11} + iJ_{21} \\
J_{11} + iJ_{21}
\end{bmatrix}
\]

We now put the beam through another polaroid with its axis horizontal. The ket vector becomes
\[
\frac{1}{2}
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
J_{11} + iJ_{21} \\
J_{11} + iJ_{21}
\end{bmatrix}
= \frac{1}{2}
\begin{bmatrix}
J_{11} + iJ_{21} \\
0
\end{bmatrix}
\]

As in C.1 or C.2, the intensity is now
\[
I_{11} = \frac{1}{4} \left[ (X_{11} - X_{21})^2 + (Y_{11} + X_{21})^2 \right]
\]
so that
\[
\frac{4I_{11} - I_2 - I_3}{\sqrt{I_2 I_3}} = 2 \sin(\theta_{11} - \theta_{21})
\]
The angle \((\theta_{11} - \theta_{21})\) is thus determined and so the differences between \(\theta_{11}\) and all the other three angles are now known.

We can regard one of the \(\theta\)'s as arbitrary, that is, we can arbitrarily set \(\theta_{11}\) to zero. This means that the angles in the other three Jones elements are now known and we have completely determined the Jones matrix.

**Example 1**

A source emits plane-polarized light of unit intensity which falls on an ideal linear polarizer. Prove that the intensity of the light coming through the polaroid is \(\cos^2 \theta\) where \(\theta\) is the angle of the polaroid measured from a position of maximum transmission. The polaroid is set to extinguish the transmitted beam and a second polaroid is then placed between the source and the first polaroid.

Show that some light can now pass through both polaroids and prove that its intensity is proportional to \(\sin^2 \phi\), where \(\phi\) is the angle of the second polaroid measured from the position of extinction.

**Solution:** Suppose the incident beam of light is plane-polarized horizontal so that its ket vector is

\[
E_1 = \begin{bmatrix} H \\ 0 \end{bmatrix}, \quad H = 1 = \text{unit intensity}
\]

The Jones matrix of the polaroid with its pass-plane at angle \(\theta\) with the x-direction is
so that the ket vector of the beam emerging from the polaroid is

\[
E_2 = \begin{bmatrix}
\cos^2 \theta & \cos \theta \sin \theta \\
\cos \theta \sin \theta & \sin^2 \theta
\end{bmatrix}
\begin{bmatrix}
H \\
0
\end{bmatrix}
= \begin{bmatrix}
H \cos^2 \theta \\
H \cos \theta \sin \theta
\end{bmatrix}
\]

The intensity of the emergent beam is given by

\[
E_2^\dagger E_2 = H^2 \cos^4 \theta + H^2 \cos^2 \theta \sin^2 \theta
\]

\[
= H^2 \cos^2 \theta \left( \cos^2 \theta + \sin^2 \theta \right)
= H^2 \cos^2 \theta
\]

Suppose we now pass the beam through another polaroid with its pass-plane vertical, so that its Jones matrix is

\[
\begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}
\]

The ket vector emerging from this polaroid will therefore be

\[
E_3 = \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
H \cos^2 \theta \\
H \cos \theta \sin \theta
\end{bmatrix}
= \begin{bmatrix}
0 \\
H \cos \theta \sin \theta
\end{bmatrix}
\]

The intensity is

\[
E_3^\dagger E_3 = H^2 \cos^2 \theta \sin^2 \theta = \frac{H^2 \sin^2 2\theta}{4}
\]

There are thus four orientations of the intermediate polaroid which give extinction.
Example 2

Elliptically polarized light is passed through a quarter-wave plate and then through a polaroid. Extinction occurs when the optic (fast) axis of the plate and the pass-plane of the polaroid are inclined to the horizontal at angles $30^\circ$ and $60^\circ$ respectively, the angles being measured in the same direction. Find the orientation of the ellipse.

**Solution:** If we substitute the given angles into the standard formulae for the Jones matrix of the polaroid and of the quarter-wave plate, we get the matrix of the polaroid followed by the quarter-wave plate

$$
\begin{bmatrix}
\cos^2 60 & \cos 60 \sin 60 \\
\cos 60 \sin 60 & \sin^2 60
\end{bmatrix}
\begin{bmatrix}
\cos^2 30 - i \sin^2 30 & \cos 30 \sin 30(1 + i) \\
\cos 30 \sin 30(1 + i) & -i \cos^2 30 + \sin^2 30
\end{bmatrix}
$$

$$
\frac{1}{4}
\begin{bmatrix}
1 & \sqrt{3} \\
\sqrt{3} & 3
\end{bmatrix}
\frac{1}{4}
\begin{bmatrix}
3 - i & \sqrt{3}(1 + i) \\
\sqrt{3}(1 + i) & 1 - 3i
\end{bmatrix}
$$

Polaroid                   Quarter-wave plate

$$
= \frac{1}{8}
\begin{bmatrix}
3 + i & \sqrt{3}(1 - i) \\
\sqrt{3}(3 + i) & 3 - 3i
\end{bmatrix}
$$

Suppose the ket vector of the entering beam is

$$
\begin{bmatrix}
H \\
Ke^{i\Delta}
\end{bmatrix}
$$
If we assign unit intensity to this beam, then in the usual notation
\[ H = \cos \theta, \quad K = \sin \theta \]
so that the incident ket is
\[ \begin{bmatrix} \cos \theta \\ \sin \theta e^{i\Delta} \end{bmatrix} \]
Since extinction is produced, the ket vector of the emerging beam from the pair of devices must have both of its elements equal to zero so that
\[
\begin{bmatrix}
3 + i & \sqrt{3}(1 - i) \\
\sqrt{3}(3 + i) & 3 - 3i
\end{bmatrix}
\begin{bmatrix}
\cos \theta \\
\sin \theta e^{i\Delta}
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
Now suppose we have a 2x2 square matrix multiplying a column matrix to produce a zero column matrix thus:
\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
X \\
Y
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
This says that \( AX + BY = 0 \) and \( CX + DY = 0 \).

Given that neither \( X \) nor \( Y \) vanishes for an elliptically polarized beam, we know at once from these equations that
\[
\frac{X}{Y} = -\frac{B}{A} = -\frac{D}{C}
\]
We have \( B = \sqrt{3}(1 - i) \) and \( A = 3 + i \) so that

\[
\frac{X}{Y} = \frac{B}{A} = \frac{\sqrt{3}(1 - i)}{3 + i} = \frac{\sqrt{3}(1 - i)(1 + i)}{(3 + i)(1 + i)} = \frac{\sqrt{3}}{1 + 2i}
\]

Therefore

\[
\frac{\cos \theta}{\sin \theta e^{i\Delta}} = \frac{X}{Y} = -\frac{B}{A} = -\frac{\sqrt{3}}{1 + 2i}
\]

and inverting

\[
\frac{\sin \theta \cos \Delta + i \sin \theta \sin \Delta}{\cos \theta} = \frac{-2i - 1}{\sqrt{3}}
\]

Equating real imaginary parts we get

\[
\tan \theta \cos \Delta = -\frac{1}{\sqrt{3}}, \quad \tan \theta \sin \Delta = -\frac{2}{\sqrt{3}}
\]

Dividing the second equation by the first now gives

\[
\tan \Delta = 2 \rightarrow \Delta = 63^\circ 26' \rightarrow \cos \Delta = 0.447
\]

The equation between the real parts now gives

\[
\tan \theta = -\frac{1}{\cos \Delta \sqrt{3}} = -\frac{1}{0.447 \times 1.732} = -1.292 \rightarrow \theta = -52^\circ 16'
\]
Example 3

A beam of right-handed circularly polarized light is passed normally through (a) a quarter-wave plate and (b) an eighth-wave plate. Both plates are to be regarded as having their optic(fast) axes vertical. Describe the state of polarization of the light as it emerges from each plate.

Solution: The circularly polarized light beam has its components H and K parallel to the two axes equal. For a right-handed state the phase difference $\Delta$ between them is $90^\circ$ so the normalized ket vector is

$$\sqrt{1} \begin{bmatrix} 1 \\ e^{i\pi/2} \end{bmatrix} = \sqrt{1} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

The Jones matrix of the quarter-wave plate with its optic(fast) axis vertical can be written

$$\begin{bmatrix} -i & 0 \\ 0 & 1 \end{bmatrix} (i) = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

so that the ket vector of the beam emerging from the quarter-wave plate is

$$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \sqrt{\frac{1}{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} = \sqrt{\frac{1}{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
This corresponds to $H=K=1$ and $\Delta = \pi$. The emerging beam is linearly polarized at an angle of $-45^\circ$ to the x-axis and lying in the second and fourth quadrants.

For the eighth-wave plate the phase shift produced between the ordinary and extraordinary waves is $45^\circ$, so that $\cos \delta$ and $\sin \delta$ are both equal to $1/\sqrt{2}$. In this case $\theta = 90^\circ$ and the Jones matrix can be written in the form

$$
\begin{bmatrix}
1 & 0 \\
0 & \frac{1}{\sqrt{2}} (1 + i)
\end{bmatrix}
\text{ or }
\begin{bmatrix}
e^{-i\pi/4} & 0 \\
0 & 1
\end{bmatrix}(e^{i\pi/4}) =
\begin{bmatrix}
1 & 0 \\
0 & \frac{1}{\sqrt{2}} (1 + i)
\end{bmatrix}
$$

so that the emerging ket vector is

$$
\begin{bmatrix}
1 & 0 \\
0 & \frac{1}{\sqrt{2}} (1 + i)
\end{bmatrix}\sqrt{\frac{1}{2}} \begin{bmatrix} 1 \\
i \end{bmatrix} = \sqrt{\frac{1}{2}} \begin{bmatrix} 1 \\
\frac{1}{\sqrt{2}} (i - 1) \end{bmatrix}
$$

From the ratios of the real and imaginary parts of the second component we see that $\tan \Delta = -1 \rightarrow \Delta = 135^\circ$, $\sin \Delta = -1/\sqrt{2} = \cos \Delta$. Inserting these values into the matrix, we see that $H=K=1$, so that using the standard form for the equation of the ellipse

$$
\frac{x^2}{H^2} - \frac{2xy}{HK} \cos \Delta + \frac{y^2}{K^2} = \sin^2 \Delta
$$

we get

$$
x^2 - \sqrt{2}xy + y^2 = 1 \quad \text{because} \quad \Delta \quad \text{lies between} \quad 0^\circ \quad \text{and} \quad 180^\circ, \quad \text{the elliptical state is right-handed.}$$
Example 4

A wave described by \( x = A \cos(\omega t + \pi / 4) \) and \( y = A \cos(\omega t) \) is incident on a polaroid which is rotated in its own plane until the transmitted intensity is a maximum.

(a) In what direction does the pass-plane of the polaroid now lie?
(b) Compute the ratio of the transmitted intensities observed with the polaroid so oriented and oriented with its pass-plane in the y-direction.

**Solution:** We first set up the ket vector of the original beam. This is

\[
A \begin{bmatrix} 1 \\ e^{-i\pi/4} \end{bmatrix} \quad \text{or} \quad A \begin{bmatrix} e^{i\pi/4} \\ 1 \end{bmatrix} \left( e^{-i\pi/4} \right) = A \begin{bmatrix} 1 \\ e^{-i\pi/4} \end{bmatrix}
\]

After passing through the polarizer at an angle \( \theta \), the ket vector becomes

\[
\begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix} A \begin{bmatrix} 1 \\ e^{-i\pi/4} \end{bmatrix} = A \begin{bmatrix} \cos^2 \theta + \cos \theta \sin \theta e^{-i\pi/4} \\ \cos \theta \sin \theta + \sin^2 \theta e^{-i\pi/4} \end{bmatrix}
\]

The intensity is given by

\[
I = A^2 \left( 1 + \frac{1}{\sqrt{2}} \sin 2\theta \right)
\]