Optical Ray Tracing Using Matrices

Reference
http://chaos.swarthmore.edu/courses/Physics50_2008/P50_Optics/03_Opt_Matrix_Meth.pdf

Theory
An optical ray at any point in a paraxial optical system can be specified by a two-element column vector in which the upper element is the distance of the ray from the optical axis (y) and the lower element is the index of refraction of the medium times the sine of the angle the ray makes with the optical axis (V)

$$\begin{bmatrix} y \\ V \end{bmatrix}.$$

Two-by-two matrices operating on this column vector then describe how the ray changes as it progresses through an optical system. If the ray traverses a distance t (measured along the optical axis) in a uniform medium of index n, then the *translation matrix* that produces the new column vector is

$$\begin{bmatrix} 1 & t/n \\ 0 & 1 \end{bmatrix}.$$

If the ray encounters a spherical boundary separating materials with indices of refraction n_1 and n_2, the *refraction matrix* that produces the new column vector after traversing the boundary is

$$\begin{bmatrix} 1 & 0 \\ -(n_2 - n_1)/r & 1 \end{bmatrix},$$

where r is the radius of curvature of the boundary. The matrix appropriate for a thin lens, which is really an approximation for two refraction and one translation matrices, can be written in terms of the focal length f of the lens

$$\begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}.$$

If the matrix is appropriate for a ray progressing from an object to the real image of the object formed by an optical system, then the matrix has a number of properties:
(a) the determinant of the matrix equals 1;
(b) the upper right element vanishes;
(c) the upper left element is equal to the transverse magnification; and
(d) the lower right element equals the inverse of the transverse magnification.

Let the distance from the object to the first reference plane of the optical system be R and the distance from the second reference plane of the optical system to the image of the object be S. Let the matrix for the optical system be

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Then the matrix appropriate for a ray traveling from the object to the real image is

$$\begin{bmatrix} 1 & S \\ 0 & 1 \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} 1 & R \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A+SC & AR+B+S(CR+D) \\ C & CR+D \end{bmatrix} = \begin{bmatrix} 1/\alpha & 0 \\ ? & \alpha \end{bmatrix},$$

where α is the inverse of the transverse magnification. Therefore, if one measures the transverse magnification for several values of R, then a graph of α vs. R has a slope equal to C and an α-intercept equal to D. Since AR+B +S(CR+D) = 0, then one can define β to be AR+B = -S(CR+D) = -Sα. Therefore, a graph of β vs. R has slope A and β-intercept B. Thus, by measuring the transverse magnification and S for several values of R, the complete matrix for the optical system (defined by what is between the two reference planes) can be determined.

Experiment

The object of the experiment is to empirically determine the matrix representing a converging and diverging lens separated by a certain distance. This matrix can then be compared to the theoretical one. To make the calculations easier, let the first reference plane be at the position of the converging lens and let the second reference plane be at the position of the diverging lens.

The first lens is a converging lens with $f_1 \approx 10$ cm. Place it on the optical bench with the bright object and use the screen to find the image it forms. By measuring the object and image distances, find the focal length of the lens using the thin lens equation.

$$\frac{1}{\ell} + \frac{1}{\ell'} = \frac{1}{f}$$

Do this for several object distances and average your values to get a final value for f_1.

The second lens is a diverging lens with $f_2 \approx -15$ cm. Measuring its focal length is a bit trickier. Set up the converging lens so a real image of the bright object forms on the screen. This real image is going to be the virtual object for the diverging lens. Place the diverging lens between the converging lens and the screen and move the screen to where the new image forms. Note the distance from the original image to the diverging lens and the distance from the new image to the diverging lens. Use these with the thin lens equation (remember virtual object distances are negative) to find the focal length of the diverging lens. Do this for several virtual object distances and average your values to a get a final value for f_2.

Now place the converging and diverging lenses 15 cm apart with the converging lens closer to the bright source. Measure both the transverse magnification (1/α) and the distance from the diverging lens to the image (S) for at least 5 values of R centered around 20 cm. Make graphs of α vs. R and β vs. R, and from these graphs determine the elements of the matrix for the optical system. Is the determinant of your matrix equal to 1 (within experimental error)?

Calculate the matrix for the optical system using the measured focal lengths and the distance between the lenses. Are the two determinations of the matrix equal (within experimental error)?