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# Field Theory for a Model of Self-Organized Criticality.

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**Abstract.** – The specific mechanism of self-organization to a critical state is identified for the Bak-Sneppen evolution model. This model is mapped exactly to an underlying branching process. Theoretical arguments, supported by numerical simulations, indicate that the resulting critical behavior is in the same universality class as Reggeon field theory.

«Self-Organized Criticality» (SOC) refers to the tendency of large dissipative systems to build up to a state poised at criticality that is characterized by a wide range of length and time scales [1]. While numerous numerical studies have found SOC to occur in specific models, there has so far been limited success in achieving an analytical understanding. This is particularly true for systems without a conserved quantity, which are probably the most common in nature. Two outstanding questions are: what is the mechanism by which self-organization occurs in a system and how is the approach to the dynamical SOC attractor characterized: secondly, what are the universal properties of the system once it has reached the critical state and what corresponding field theories (if any) describe those properties. Here we answer these questions for a specific dynamical system.

Recently, Bak and Sneppen [2] introduced a particularly simple model (BS) as a coarse-grained description of evolution. This model exhibits robust SOC behavior; the criticality is insensitive to the particular details of the model. Within a mean-field framework, Flyvbjerg *et al.* [3] showed that the critical state is an attractor for the dynamics and determined critical exponents. A similar model was constructed by Zaitsev [4] to describe low-temperature creep phenomena such as dislocation glide. Here, it is shown that the BS model can be understood entirely in terms of a specific branching process, the BS branching process, which describes the dynamics of activity in the system. We explicitly describe how the branching process modifies the global environment in which it propagates until the branching process itself becomes critical. Concurrently, the environment achieves stationarity at the SOC attractor.

Symmetry arguments and numerical simulations suggest that the critical exponents for the BS model can be derived from Reggeon field theory (RFT) [5]. Perhaps the best known example of the RFT universality class is directed percolation (DP) [6]. For the BS model, the self-organization process, *i.e.* the relaxation to the critical attractor, is characterized by an exponent $\gamma - 1$, where $\gamma$ is the susceptibility exponent for DP. The distribution of avalanches of size $s$ is a power law, $N(s) \sim s^{-\tau+1}$, where the exponent $\tau$ is that of the cluster size distribution in DP [7]. Similarly, the anomalous diffusion of activity is given by $r \sim s^h$, where $r$ is the spatial width of activity, and $h = (d_f + z)^{-1}$. Here, $z$ is the dynamical critical exponent relating space and time, or $z = \nu_\parallel/\nu_\perp$ in DP. $d_f$ is the fractal dimension of the active front in DP, *i.e.* $n_{act} \sim r^{d_f}$, where $n_{act}$ is the number of active sites at a given time. An unexpected result of these considerations is that the critical exponents for the «Game of Life» model may also be derived from RFT in two dimensions.

The BS model is defined as follows. Random numbers $p_i$ are assigned to each site of a $d$-dimensional lattice. Initially, the random numbers are selected from a flat distribution: $\mathscr{P}(p_i) = 1$ for $0 \leqslant p_i \leqslant 1$, and $\mathscr{P}(p_i) = 0$ otherwise. At each time step, the lowest random number is located; then the random numbers at that minimal site and at each of its $2d$ nearest neighbors are replaced with new random numbers which are also drawn from $\mathscr{P}$. After many minimal sites have been selected and updates performed, the $p_i$ on the lattice approach a stationary distribution which differs from $\mathscr{P}$. Numerical simulations show that the density of sites with $p_i$ less than a threshold, or gap, $p_c$ vanishes, while the density above the gap is uniform [2]. Bak, Sneppen, and Flyvbjerg [2, 3] found that the activity takes place in terms of intermittent bursts, or avalanches. Numerical simulations on a one-dimensional model revealed a power law distribution of avalanche sizes, with the characteristic exponent $\tau - 1$ around 1.0 [2]. The spreading of activity within an avalanche was found to be subdiffusive, $h \approx 0.4$ [8].

We first address the mechanism of self-organization to the critical state. Initially, the minimal $p_i$ is selected, $p_{min}(0)$, and $2d + 1$ new random numbers are selected. The quantity $p_{min}(0)$ is the initial value of the gap. If all of the new random numbers chosen are greater than $p_{min}(0)$, then at the next time step the minimal site in the system must have a random number which is greater than $p_{min}(0)$, so that $p_{min}(1) > p_{min}(0)$, and the gap in the system has widened. If, however, one or more of the $(2d + 1)$ new random numbers is less than $p_{min}(0)$, then those sites which were activated by the initial minimal site must be the next activated, and so on. $p_{min}$ cannot increase until all sites activated with $p_i < p_{min}$ are eliminated through subsequent selection and update.

The gap, $p_{min}(s)$, is the largest of the minimal $p_i$ which have been selected up to time $s$, starting from an initial distribution given by $\mathscr{P}$. The gap is a monotonically increasing function of time with flat plateaus that become larger and larger. The size of an avalanche is the number of time steps separating events where the gap jumps to its next, higher value, *i.e.* the width of a plateau. An individual avalanche can be mapped exactly to a specific realization of the BS branching process as follows: 1) during an avalanche, or branching process, all sites for which $p_i < p_{min}(s)$ are denoted as active sites or *particles*; a state with no active sites is denoted the *vacuum*. 2) An avalanche is started with a single particle. This occurs when the gap has just jumped to a new value. In a given branching process particles are created with probability $p_{min}$, given by the minimal site that started the new branching process. 3) Each of the particles carries its own random number $p_i < p_{min}$. The one with the minimal number is selected next, and can create particles at its $2d + 1$ allowed sites with probability $p_{min}$. 4) The branching process continues until the vacuum is reached. Then a new minimal number is chosen somewhere else in the system, and the gap jumps on average by an amount $(1 - p_{min})/L^d$, where $L$ is the size of the system. Then a new branching process has started with this new, higher value of $p_{min}$.

The growth of the gap *vs.* time $s$ is described by the following equation:

$$\frac{\partial p_{\min}}{\partial s} = \frac{1 - p_{\min}}{L^d \langle s \rangle_{p_{\min}}} . \tag{1}$$

The quantity $\langle s \rangle_p$ is the average number of events in the BS branching process for a given branching probability $p$. As the gap increases, the average size of the branching process also increases, and eventually diverges as $p_{\min} \to p_c$. At this point, the BS branching process is critical, and the distribution of $p_i$ in the system achieves stationarity, where in the thermodynamic $L \to \infty$ limit, the density of sites with $p_i < p_c$ vanishes, and the distribution of $p_i$ is uniform above $p_c$. Equation (1) defines the mechanism of SOC for the BS model. It maps the transient behavior of the model to the behavior of the branching process below criticality.

In order to solve eq. (1), we need to determine precisely how the size $\langle s \rangle_p$ diverges as the critical state is approached. This divergence is governed by the fixed point which describes scaling at and near criticality, to which we now turn.

In the critical state, an avalanche may be defined as the number of events separating instances when the minimal particle chosen is within $\delta p$ of $p_c$. Since at each time step, the particle with the lowest $p_i$ is chosen, the values of $p$ among particles are correlated, while values of $p$ among the vacuum sites are not. This strict lack of correlations among vacuum sites makes this model analytically tractable. In particular, all properties of the BS model can be studied by examining the exactly equivalent branching process, as described above.

In order to proceed further, we, for the moment, ignore correlations among particles, and consider the particles involved in the branching process to carry random numbers which are *annealed* at each step. This latter branching process, denoted temporal percolation, is that of RFT. (In the actual BS branching process, of course, these random numbers are quenched.) In the continuum description, symmetry arguments, outlined below, indicate that the local density of particles $n(x, t)$ at time $t$ (1) and position $x$ in the branching process evolves according to

$$\frac{\partial n(x, t)}{\partial t} = -\Gamma( - \nabla^2 n(x, t) + rn(x, t) + un(x, t)^2) + \eta(x, t) . \tag{2}$$

The correlations of the Gaussian random noise with zero mean obey

$$\langle \eta(x, t)\, \eta(x', t') \rangle = 2Dn(x, t)\, \delta^d(x - x')\, \delta(t - t') . \tag{3}$$

These RFT equations describe a dynamical transition in a noisy system between a state with macroscopic activity in the long-time limit ($n > 0$), and an absorbing inactive state at $n = 0$. They are the simplest local equations, which are isotropic in space and translationally invariant in space and time, that describe a branching process where interacting particles can wander, die, annihilate, or create new particles [9-11]. The noise amplitude vanishes when $n = 0$, consistent with the vacuum being a true absorbing state from which the system cannot escape. Since the noise acts independently on each particle, in a region with $n$ particles the total noise is simply the sum of $n$ independent random variables. Consequently, the noise amplitude is proportional to $n$.

The first term on the right-hand side of eq. (2) describes the diffusive wandering of

(1) The variable $t$ corresponds to a rescaled time in the simulation, *i.e.* at each step $s \to s + 1$, while $t \to t + 1/n_{\text{act}}(t)$.

particles. The second term describes the tendency of the number of particles to grow or decay. In mean-field theory, there is a transition at $r = 0$ where the absorbing state becomes unstable, and a finite density of particles appears in the steady state. The nonlinear term describes interactions between particles. It is allowed because the order parameter is nonnegative, so that the equation is not required to have $n \to -n$ symmetry. In principle higher-order terms in $n$ and its derivatives are present; standard renormalization group arguments show that these are irrelevant near the upper critical dimension, which is $d_{\mathrm{uc}} = 4$.

The effect of the correlations in the BS model is to change the sequence of updating particles. The number of particles in an on-going branching process can be affected by these correlations only when a particle annihilates a particle that is located at a nearest-neighbor site. This leads us to suggest that the BS model differs from temporal percolation only in the value of coarse-grained coefficients, *e.g.* $\Gamma$, $r$, $u$, $D$, of local terms. We note that nonequilibrium critical points in single-component reaction diffusion systems are generically described by eqs. (2), (3) [9], using similar symmetry arguments. A simple example is the contact process [12], which is a lattice version of Schøgl's first model [13]. Numerous studies have demonstrated the robustness of RFT critical behavior in spite of dramatic differences in the evolution rules used and different correlations for the dynamics [14].

We are now in a position to evaluate the process of self-organization to the critical state, eq. (1). Near $p_c$ the average avalanche size diverges $\langle s \rangle_p \sim (p_c - p)^{-\gamma}$. Equation (1) can be integrated to give

$$p_c - p \sim (s/L^d)^{-\rho}, \qquad \rho = 1/(\gamma - 1). \tag{4}$$

This relation is valid irrespective of the nature of the underlying fixed point. It holds over the range $L^d \ll s \ll L^{\tilde{d}}$, where $\tilde{d} = d + (\gamma - 1)/\nu_\perp$. The lower limit requires that the avalanches are in the scaling regime; while the upper limit requires that the cut-off in the spatial extent of the avalanches be much less than the system size. If the fixed point for the BS model is described by RFT, as we suggest, then the relaxation to the fixed-point value of the gap *vs.* the «time» or total number of events is given by the exponent $\rho \approx 0.78$ [15] in one dimension. The exponent for the distribution of avalanches for $d = 1$ is $\tau - 1 \approx 1.11$ [15], and the diffusion exponent $h = (z + d_f)^{-1} \approx 0.430$ [15], also in one dimension.

In order to efficiently measure the critical exponents numerically, we simulated the *actual* BS *branching process*, where the random numbers are quenched, starting from a single particle in the vacuum state. This means that, in the simulation, we always choose the particle with the smallest random number to be the next site updated. We do not numerically study the annealed model.

Varying the branching probability $p_{\min}$ introduces a tunable control parameter, which corresponds to the gap in the BS model, that allows measurements both at criticality and at the transient states. This method also eliminates all finite-size effects. All simulations were performed in one dimension (2). Figure 1 shows the spatial width, $r$, of the cluster *vs.* $s$, near $p = p_c$. The asymptotic slope yields a diffusion exponent $h \approx 0.431$ in fair agreement with ref. [8] and excellent agreement with the value from DP. We found that this value of $h$ was extremely robust on varying $p$ near $p_c$. This exponent is our most precise measurement and also agrees best with the predicted value. Figure 2 shows the distribution of avalanches at

(2) After submission of this paper, we have performed extensive simulations in higher dimensions, to be presented elsewhere. All of our results are in agreement with the hypothesis that the BS model is in the RFT universality class.

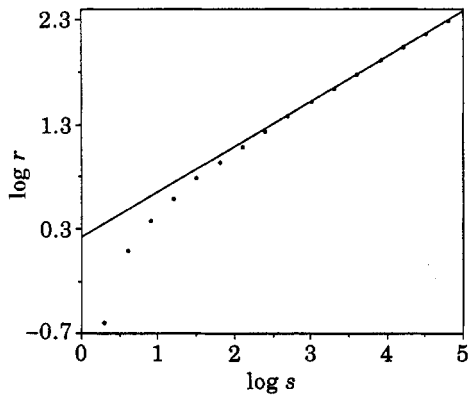


Fig. 1.

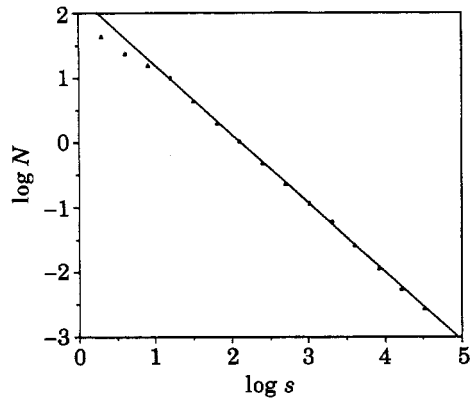


Fig. 2.

Fig. 1. – Cluster width $r$ *vs.* time $s$ for the branching process at $p = 0.667$. The straight line indicates an asymptotic slope $h \approx 0.431$.

Fig. 2. – Distribution of avalanches in the branching process for $p = 0.667$. The straight line indicates an asymptotic distribution, $N(s) \sim s^{-1.08}$.

the critical point $p_c \approx 0.667$ giving $\tau - 1 \approx 1.08$ in agreement with the result from RFT. We suspect that the critical value may be $p_c = 2/3$. The value of $p_c$, and the value of $\gamma$, was numerically determined by estimating the divergence of the average cluster size both above and below $p_c$. The apparent value of $\tau$ changes considerably on varying $p$ near $p_c$, from $\tau \approx 1.00$ at $p = 0.666$ to $\tau \approx 1.16$ at $p = 0.672$, and leads to a rather large uncertainty ($\pm$ 0.04) in our determination of $\tau$. Similar effects have been seen in other one-dimensional systems belonging to the RFT universality class [10]. Our attempt to determine the exponent $\gamma$ characterizing the average size of the avalanches *vs.* $(p - p_c)$ was hindered by the large variation in apparent $\tau$ and consequently a very small asymptotic critical region where the distribution approaches a fixed power law with an exponential cut-off. Our best attempt, involving approximately $10^{11}$ updates, gave $\gamma \approx 2.3$ compared with $\gamma \approx 2.28$ from DP, but we cannot provide accurate error bars for $\gamma$ given the limits of our computational capability.

The BS model was introduced as an attempt to describe the evolution of an ecology of interacting species. The present theory relates the model to RFT, which has also been used to describe autocatalytic chemical processes. This vaguely suggests that there might be a connection between biological life and catalytic processes. This speculation has been arrived at from a very different perspective by, among others, Kauffman [16].

The Game of Life is a two-dimensional cellular automaton which has been found to be at or very near criticality [17]. Within a coarse-grained description, the dynamics should be given by RFT in two dimensions, with exponent $\tau - 1 \approx 1.27$ for the distribution of avalanche sizes and $\tau_t - 1 \approx 1.45$ for avalanche durations. Recent large-scale simulations by Hemmingsson yield $\tau - 1 \approx 1.22$ [18]; Alstrøm obtains $\tau - 1 \approx 1.27$ and $\tau_t - 1 \approx 1.41$, respectively [19]. Finally, we suspect that Wolfram's [20] classification of cellular automata may correspond to the subcritical, critical and supercritical regimes of RFT.

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