Random Phase Approach to Quantum Measurement Problem

Feng and Chen

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Abstract

A random phase approach to the quantum measurement problem is developed. According to the concept of decoherence, in a quantum measurement the observed system is open to a macroscopic apparatus. This open system can be well described by some kind of “almost” quotient Hilbert space formed phenomenally according to some stability conditions. That is, the relevant Hilbert space can be classified into some equivalent classes, for which the superposition principle is not suitable. Moreover, a group of random phase unitary operators can be introduced so that the “almost” quotient space is reduced to an effective quotient space. In the effective quotient space, a density operator can be constructed by integrating over the random phases’ space, leading to a statistical ensemble, or the Born’s probability rule. In an analogous way, a mechanism is given to explain why the states of a macroscopic system behave more classically, which plays a crucial role in the quantum measurement. In the end, a general description of a measurement process is developed, showing that the definite outcomes are actually caused by the “almost” quotient Hilbert space of the macroscopic apparatus.

1. Introduction

Quantum measurement is an important problem to understand quantum mechanics, in particular it is significant to reconcile the microscopic quantum world with the macroscopic classical world perceived by us. The key distinction is the superposition principle in quantum mechanics, which says that any linear combination of a system’s states are still a possible state of that system. But this principle seems not to be suitable for the ordinary macroscopic systems perceived by us. For example in a measurement, the outcomes that the apparatus records must be definite, not a superposition of them, indicating that the state of the observed system will also become definite after the measurement, i.e., the problem of definite outcomes. In the familiar Copenhagen Interpretation, this is explained by a so called “state collapse” of the observed system, a mysterious process.

Another explanation is proposed by quantum decoherence [1,2]. The key idea is that realistic quantum systems are never isolated, but are immersed in the surrounding environment and interact continuously with it. Then, “decoherence is caused by the interaction with the environment which in effect
monitors certain observables of the system, destroying coherence between the pointer states corresponding to their eigenvalues" [1]. An environment-induced superselection or einselection scheme [3] has been developed to handle the measurement problem, saying that the apparatus’s pointer states are determined or selected by the form of the interaction between the apparatus and its environment.

The einselection is briefly as follows. Suppose the observed system’s initial state is $\alpha |S_1\rangle + \beta |S_2\rangle$. After a measurement, we obtain a correlated state for the combined system-apparatus

$$\alpha |S_1\rangle |A_1\rangle + \beta |S_2\rangle |A_2\rangle$$

(1)

There is a basis ambiguity that we can rewrite the above correlated state in any other basis, the so called preferred-basis problem. For example, we could combine the apparatus’s states $\{|A_1\rangle, |A_2\rangle\}$ linearly to obtain another couple $\{|A_1'\rangle, |A_2'\rangle\}$ so that the corresponding states for the system $\{|S_1'\rangle, |S_2'\rangle\}$ may not be the eigenstates of the measured observable. This is possible in principle due to the superposition principle. The einselection scheme tries to resolve this problem by coupling the apparatus with an environment, obtaining a more correlated state

$$\alpha |S_1\rangle |A_1\rangle |E_1\rangle + \beta |S_2\rangle |A_2\rangle |E_2\rangle$$

(2)

Moreover, a stability criterion $[\hat{O}_A, \hat{H}_{A,E}] = 0$ was suggested by Zurek [3] so that the system-apparatus correlations are left undisturbed by the subsequent formation of correlations with the environment, i.e., there is a preferred apparatus observable. Then provided $(E_1 | E_2) \nsim 0$, a reduced density operator for the system-apparatus can be obtained by performing a partial trace over the environment’s space or decoherence from the environment

$$|\alpha|^2 |S_1\rangle \langle S_1| |A_1\rangle \langle A_1| + |\beta|^2 |S_2\rangle \langle S_2| |A_2\rangle \langle A_2|$$

(3)

which is a statistical ensemble with definite outcomes, realizing the Born probability rule for the observed system. Although the einselection scheme gives the required results in Eq. (3), there are still some problems with this scheme as follows:

(i) In deriving the Born rule Eq. (3), a partial trace has been used, which seems to be a circular argument [1, 2] since the partial trace itself
can also be treated as one part of the Born rule. In other words, a partial trace over the environment’s space can be regarded to be another measurement on the environment. For this reason, Zurek introduced the concept of envariance [1, 3-5], i.e. environment-assisted invariance to give another derivation of the Born rule, but his method is not general enough to be used wildly.

(ii) According to the einselection scheme, the apparatus’s pointer states are selected or determined by the form of the interaction between the apparatus and its environment. It seems that the interaction from the environment plays a dominate role in the measurement process. However, from Eq. (1) to Eq. (3) there is an implicit order that the interaction between the system and the apparatus is first turned on. Although in the real case the two interactions actually occur independently, the implicit order indicates that the interaction from the environment can only be a perturbation. In fact, another stability criterion $[\hat{O}_S, \hat{H}_{S,A}] = 0$ can be proposed for the observed system, which can remove the basis ambiguity in Eq. (1) effectively in an analogous way as $[\hat{O}_A, \hat{H}_{A,E}] = 0$ in the einselection scheme.

(iii) The orthogonality condition $\langle E_1 | E_2 \rangle \approx 0$ for the environment’s states are not satisfied generally, though in some controlled model this condition can be achieved approximately. In a more general case, the environment is inaccessible and uncontrolled so that the orthogonality condition can not be proposed without any question. In other words, the partial trace can not always be treated as an effective method to induce a decoherence. Some new method is needed so that the required condition can be proposed for general macroscopic systems.

(iv) The einselection scheme does not provide an effective description of macroscopic systems. For example, for a macroscopic apparatus in the quantum measurement, all the scheme says is that the apparatus decoheres in some basis due to the interaction with its environment. In fact, to the observed system, the apparatus can also be treated as an environment, since it is also a macroscopic system containing large number of degrees of freedom. In other words, the pointer states $\{|A_1\rangle, |A_2\rangle \}$ in Eq. (1) stand for only a very small portion of the apparatus, with the rest degrees of freedom ignored. This ignorance can also be described by a partial trace, leading to a reduced density operator without an
extra environment. In this sense, the classical properties of an ordinary macroscopic system may also result from a partial trace over its unobserved degrees of freedom. Thus, the new method to induce a decoherence should also provide an effective description of macroscopic systems.

(v) Generally speaking, decoherence provides a coarse grained description to an open system interacting with its environment, by tracing over the large number of unobserved degrees of freedom of the environment. The reason for this procedure is our inability to keep track of all the degrees of freedom of the environment. In this sense, decoherence seems to be only an approximative method for our human beings to describe an open system. However, a partial trace seems to be a more "general" operation acting on a Hilbert space. For example, we can simply perform a partial trace over one qubit’s space on a Bell state, without involving the above reason apparently. Thus, the new method to induce a decoherence should also demonstrate the reason for introduction of decoherence apparently.

For a large closed system, suppose one small portion or subsystem is singled out to be studied or observed, then the rest subsystems can serve as an environment. More precisely, these can be expressed as

\[ \mathcal{H}_t = \mathcal{H}_S \otimes \mathcal{H}_E \] (4)

with \( \mathcal{H}_t, \mathcal{H}_S \) and \( \mathcal{H}_E \) the Hilbert spaces of the total closed system, the studied system and the environment respectively. For the partial trace approach to decoherence, the evolution of the studied system can be described by a superoperator acting on an initial density operator [6]. The superoperator is obtained from some total unitary operator \( \hat{U}_t \) by tracing over the environment’s space. This gives a coarse grained description to the studied system, since quantum information stored in the correlations with the environment is “lost” during the partial trace. The meaning for “lost” here is due to the ignorance of the large number of degrees of freedom of the environment. This ignorance is expressed apparently, since after the partial trace we are working only on a smaller Hilbert space \( \mathcal{H}_S \) effectively.

However, as is well known, the information is in fact not lost from the view of the total closed system. To demonstrate this fact in an obvious way, we’d
better to work on the total Hilbert space \( \mathcal{H}_t \). Then how to obtain a coarse grained description of the studied system? Recall that the total Hilbert space is closed under all the unitary transformations. These transformations can be roughly divided into two classes: one class contains the non-dynamical representation transformations that result from some coordinate transformations; while the other one contains all the dynamical unitary evolutions. Those representation transformations can be neglected by choosing a particular representation or gauge. After that, we can just consider the dynamical evolutions. For a closed system, the total Hilbert space can be *ergodic*\(^1\) from one particular state under all the possible evolutions, at least in principle. In other words, the total Hilbert space is *fine grained* under the “detection” of those evolutions. Then if we apply a smaller subset of evolutions satisfying some condition, the fine structure of the total Hilbert space may not be “detected” so that only a *coarse grained* description can be obtained. In this paper, we shall show that both a quantum measurement process and a macroscopic system can be described in this way, without using a partial trace.

The paper is organized as follows. In Sec. 2, an “almost” quotient space is formed to describe a particular open system, an observed system open to a measurement apparatus. This “almost” quotient space and a following random phase approximation provide a coarse grained description to the observed system. Then in Sec. 3, by means of a random phase unitary operator, an effective quotient Hilbert space is obtained from the “almost” quotient space to describe the observed system. In that quotient space, the superposition principle is not proper, and a density operator for a statistical ensemble is obtained to give the Born rule without using a partial trace. A two-dimensional example, the Stern-Gerlach experiment is analyzed in Sec. 4 to give a first look at a measurement process. After that, we give a mechanism for structure formation of a macroscopic system, also by forming an “almost” quotient space, so that the macroscopic system?*s* states behave more classically. A general description of a quantum measurement process is proposed in Sec. 6, with the problem of the Schrödinger’s cat also resolved by treating it as a particular measurement problem. Finally, in Sec. 7, we

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\(^1\)In mathematics, the term *ergodic* is used to describe a dynamical system which, broadly speaking, has the same behavior averaged over time as averaged over the space of all the system’s states (phase space). In physics the term is used to imply that a system satisfies the *ergodic* hypothesis of thermodynamics.
summarize our results and draw some conclusions.

2. “Almost” Quotient Hilbert Space

For the total Hilbert space $\mathcal{H}_t$ in Eq. (4) that is roughly resolved into a studied or observed system and an environment, it is *ergodic* from one particular state under all the possible dynamical evolutions. The number of those evolutions is large and can be collected as a set or a group $\{\hat{U}\}$ for short. Under this group, $\mathcal{H}_t$ is “detected” in a *fine grained* way. Then, consider an observable $\hat{O}$ for the studied system with an expansion in terms of orthonormal eigenstates

$$\hat{O} = \sum_n O_n \langle n | \hat{\psi} \rangle, \quad \hat{O} | n \rangle = O_n | n \rangle$$

(5)

If performing a quantum measurement on a *fixed or time independent* normalized state $|\psi\rangle$ on $\mathcal{H}_S$, we will obtain an average value

$$\bar{O} = \langle \psi | \hat{O} | \psi \rangle = \sum_n O_n |\alpha_n|^2, \quad |\psi\rangle = \sum_n \alpha_n |n\rangle$$

(6)

with $|\alpha_n|^2$ treated as the probability for obtaining the state $|n\rangle$ after the measurement. If the operator $\hat{O}$ is an identity $\hat{I} = \sum_n |n\rangle \langle n |$, then we have

$$1 = \langle \psi | \hat{I} | \psi \rangle = \langle \psi | \psi \rangle = \sum_n |\alpha_n|^2$$

(7)

which is the sum rule of probability. These are the standard results of quantum mechanics, named Born rule.

Now let’s consider the problem in a different way. For Eq. (7), it is invariant under the group $\{\hat{U}\}$ of all the evolutions, since $[\hat{U}, \hat{I}] = 0$. This is also a standard result of quantum mechanics. However for Eq. (6), it is invariant only under some particular unitary evolutions, which can be collected as a smaller group

$$G_{\hat{O}} = \{\hat{U}, [\hat{U}, \hat{O}] = 0\}$$

(8)

The condition $[\hat{U}, \hat{O}] = 0$ in Eq. (8) is a stability condition for the observed system, which guarantees that the measurement can be accomplished without uncontrolled disturbances. In a quantum measurement, the macroscopic apparatus serves as an environment, thus we can first consider this particular case, with $\mathcal{H}_E$ in Eq. (4) replaced by the apparatus’s Hilbert space $\mathcal{H}_A$. 6
Then the evolutions in the group $G_\hat{O}$ stand for the interactions between the observed system and the large number of degrees of freedom of the apparatus. Certainly, there may be some perturbations that can relax the stability condition to $[\hat{U}, \hat{O}] \approx 0$, but this does not change the main results. In a word, the group $G_\hat{O}$ should be treated as a necessary prerequisite to measure the observable $\hat{O}$. Then what is its effect on the observed system?

Act the group $G_\hat{O}$ on the observed systems space, or more precisely $\mathcal{H}_S \otimes \mathcal{H}_A^0$ in which the state is of some direct product form $|\psi\rangle |\phi\rangle_A^0$ with $|\psi\rangle$ and $|\phi\rangle_A^0$ the initial states of the system and apparatus respectively. Then we will obtain a subspace $\mathcal{H}'$ of $\mathcal{H}_t$, since $G_\hat{O}$ is only a subgroup of the total one $\{\hat{U}\}$. There is a special property in $\mathcal{H}'$ that this space is classified into some equivalent classes according to the group $G_\hat{O}$. For instance, let any two different evolutions $\hat{U}_1$ and $\hat{U}_2$ of the group $G_\hat{O}$ act on some initial state, then we have

$$|\psi\rangle |\phi\rangle_A^0 \cong \hat{U}_1 |\psi\rangle |\phi\rangle_A^0 \cong \hat{U}_2 |\psi\rangle |\phi\rangle_A^0$$

(9)

This can be easily verified by substituting Eq. (9) into Eq. (6), by noting that $[\hat{U}_{1,2}, \hat{O}] = 0$. More precisely, we can express the space $\mathcal{H}'$ formally as

$$\mathcal{H}' \equiv G_\hat{O}(\mathcal{H}_S \otimes \mathcal{H}_A^0) = \{ \{G_\hat{O} |\psi\rangle |\phi\rangle_A^0 \}, \{G_\hat{O} |\varphi\rangle |\phi\rangle_A^0 \}, \ldots \}$$

(10)

as long as the expansion coefficients of the fixed states of $\mathcal{H}_S |\psi\rangle = \sum_n \alpha_n |n\rangle$ and $|\varphi\rangle = \sum_n \beta_n |n\rangle$ satisfy

$$|\alpha_n|^2 \neq |\beta_n|^2 \quad \text{for some } n$$

(11)

In a real measurement, there should be a sample of prepared states of the observed system, one measurement for each prepared state. Then the initial state of the apparatus for each measurement should be “almost” the same so that no significant disturbance occurs. Here, we fix the apparatus’s initial state for simplicity without changing the main results.

The complementary subspace $\mathcal{H}_t - \mathcal{H}'$ can be arrived at only by those evolutions not belonging to the group $G_\hat{O}$, that is, by some evolutions noncommutative with the observable $\hat{O}$. In this way, the total Hilbert space $\mathcal{H}_t$

$$\mathcal{H}_t = \mathcal{H}' \oplus (\mathcal{H}_t - \mathcal{H}') \cong \mathcal{H}_t / G_\hat{O}$$

(12)

is “detected” by those evolutions of the group $G_\hat{O}$ only in a coarse grained way. The meaning of “coarse grained” is as follows. The equivalent class
\( \{ G_{\hat{O}} | \psi \} | \phi \rangle \) in Eq. (10) can be treated as a “state” in the space \( \mathcal{H}' \), while the fine structure of the class is “ignored” due to our inability to know the details of those evolutions in the group \( G_{\hat{O}} \). The classification in Eq. (12) is not an exact quotient space due to the complementary subspace \( \mathcal{H}_t - \mathcal{H}' \), so we call it an “almost” quotient space. The quotient space notation \( \mathcal{H}_t / G_{\hat{O}} \) is used here only to denote this classification of \( \mathcal{H}_t \) for short, similarly for some other \( \mathcal{H}_t / G() \) in the following discussions, with \( G() \) a possible subgroup of unitary evolutions. Since the environment here is only a measurement apparatus, thus we can concern only with \( \mathcal{H}' \), otherwise no credible measurement could be performed.

The representative state for one equivalent class in the space \( \mathcal{H}' \) corresponding to some fixed state \( \sum_n \alpha_n |n\rangle \) in \( \mathcal{H}_S \) can be expressed as

\[
\sum_n \alpha_n |n\rangle |\chi_n\rangle_A , \quad \sum_n |\alpha_n|^2 = 1
\]  

(13)

with \( \{|\chi_n\rangle_A\} \) a collection of some normalized (not necessary orthogonal) states for the apparatus. We can further make an expansion for each \( |\chi_n\rangle_A \)

\[
|\chi_n\rangle_A = \sum_{i,j,k,...} C_{n; i, j, k,...} |i, j, k, \cdots\rangle_A
\]  

(14)

with \( i, j, k, \cdots \) denoted as the large number of degrees of freedom of the apparatus. Here we only give some general considerations, while a more detailed discussion with the apparatus’s pointer states singled out will be shown in Sec. 6. There is also a possible basis ambiguity by transforming the set of states \( \{|\chi_n\rangle_A\} \) into another set. As a consequence, the set of the observed system’s states will also be changed. For instance, by substituting Eq. (14) into Eq. (13), we have

\[
\sum_{i,j,k,...} \left( \sum_n \alpha_n C_{n; i, j, k,...} |n\rangle \right) |i, j, k, \cdots\rangle_A
\]

where the linear combination in the bracket is a new set of states for the observed system. This basis ambiguity is always present in principle for a closed system’s state, just like the case in Eq. (1). However, the ambiguity here is just a non-dynamical gauge freedom, since if it had some dynamical sources, the involved evolutions would violate the stability condition \([\hat{U}, \hat{O}] = 0 \) in the group \( \{ G_{\hat{O}} \) so that no credible outcomes could be obtained. Thus,
we can fix the gauge by choosing the expansion’s form to be the one in Eq. (13), removing the basis ambiguity.

The quantum information is stored in the correlations of the system and the apparatus, in particular in the coefficients \( \{ C_{n;i,j,k,\ldots} \} \) of Eq. (14)

\[
|C_{n;i,j,k,\ldots}| \exp (-i\gamma_{n;i,j,k,\ldots})
\]

The singled out phase factor gives the phase information exchanges between the observed system and the apparatus’s degrees of freedom \( i, j, k, \ldots \). In the large number of the apparatus’s degrees of freedom limit, the phase in Eq. (15) can be approximated to be random depending only on the quantum number \( n \)

\[
\alpha_n e^{-i\gamma_{n;i,j,k,\ldots}} \rightarrow \alpha_n e^{-i\gamma_n} \rightarrow |\alpha_n| e^{-i\gamma_n}
\]

where the last step in Eq. (16) indicates that the phase information stored in each \( \alpha_n \) has been randomized. This random phase approximation concerns only with the total effects of the apparatus on the observed system, ignoring the details of the interactions. Certainly, the reason for this approximation is due to our inability to keep track of the apparatus’s large number of degrees of freedom. This implies that the random phase approximation can be treated as a further coarse grained approximation by ignoring some details of the information exchanges. Moreover, this approximation depends little on the states \( \{ |\chi_n\rangle_A \} \) of the apparatus, while a partial trace needs a further condition that those states should be orthogonal. After the random phase approximation information is lost partially, just like the case of a partial trace in the einselection scheme.

It should be stressed that the full information is actually not lost, but still stored in those correlations implicit in Eq. (13), in particular in those combined coefficients

\[
\{ \alpha_n C_{n;i,j,k,\ldots} \equiv D_{n;i,j,k,\ldots} \}
\]

Comparing a state \( \sum_n \alpha_n |n\rangle \) in the space \( \mathcal{H}_S \) with its corresponding equivalent class in the space \( \mathcal{H}' \) we can see that, as an open system the information of the system’s initial state stored in \( \{ \alpha_n \} \) has diffused into the correlations \( \{ D_{n;i,j,k,\ldots} \} \) with its surrounding environment, the apparatus here. To recover the exact information of the system’s initial state, we should know the precise details about the information exchanges among those coupled systems. This seems to be impossible for any ordinary observer including our human
beings, may only be possible for some kind of “superobserver”. For ordinary observers, the random phase approximation is thus introduced to obtain an effective coarse grained description. We shall show in Sec. 6 that this random approximation is also necessary for some macroscopic observations on the apparatus made by our human beings, since our not developed brains cannot deal with the details of the information exchanges.

The above discussions imply that, when making a quantum measurement, the large number of degrees of freedom of the apparatus leads to various evolutions in the group $G_{\hat{O}}$ and results in a classification of the total Hilbert space, as given in Eq. (12). This classification and a further random phase approximation provide a coarse grained description to the observed system. For another observable $\hat{O}' = \sum_i O'_i |i\rangle \langle i|$ that satisfies $[\hat{O}', \hat{O}] \neq 0$, there will be another classification of the space $\mathcal{H}_t$ according to some group $\{G_{\hat{O}'}, \hat{O}'\}$. In this sense, the difference of measurement outcomes for two noncommutative observables comes from the fact that the relevant Hilbert space is classified in two different ways.

Now, let’s consider a measurement on a time dependent state $|\psi(t)\rangle$ evolved via some evolution $\hat{U}(t)$ which results from some other environment. In this case, an extra apparatus should be added to make a measurement. If $\hat{U}(t)$ is commutative with the observed observable $\hat{O}$, then it belongs to the group $G_{\hat{O}}$ and the above discussion is unchanged. If $\hat{U}(t)$ is noncommutative with the observable $\hat{O}$, then the Hilbert space will be some evolutionary one $\hat{U}(t)(\mathcal{H}_S^0 \otimes \mathcal{H}_E^0)$ from some particular initial space. At some instant, we can make a measurement by turning on the evolutions in the group $\{G_{\hat{O}}\}$, obtaining another $\mathcal{H}'$ formally expressed as $\{G_{\hat{O}} \hat{U}(t)(\mathcal{H}_S^0 \otimes \mathcal{H}_E^0 \otimes \mathcal{H}_A^0)\}$. Then the mean value of the observable is given by

$$\bar{O}(t) = \langle \psi(0) | (\hat{U}\hat{U})(t)^\dagger \hat{O}\hat{U}(t) | \psi(0) \rangle = \langle \psi(t) | \hat{O} | \psi(t) \rangle$$  \hspace{1cm} (17)

with some arbitrary $\hat{U}$ chosen from the group $G_{\hat{O}}$. This means that when making a (real) measurement, the evolution $\hat{U}(\tau)(\tau > t)$ should be stopped, otherwise no credible outcomes can be obtained. This is crucial for a quantum measurement, since the strengths of the measurement interactions or those evolutions of the group $G_{\hat{O}}$ are roughly equal to that of $\hat{U}(t)$. In this sense, continuous measurement is forbidden, and one measurement gives one outcome for a sample of prepared states. While for a (classical) observation on a macroscopic system, for example our sense of vision, the observation is
only a small perturbation to the evolution of the macroscopic system so that continuous observation is possible.

One prepared state $|\psi(t_0)\rangle$ at some instant $t_0$ can be expanded as

$$\sum_n \alpha_n(t_0) |n\rangle |\chi_n(t_0)\rangle_E , \quad \sum_n |\alpha_n(t_0)|^2 = 1 \quad (18)$$

with $\{|\chi_n(t_0)\rangle_E\}$ still a collection of some normalized (not necessary orthogonal) states for the environment. Analogous to Eqs. (15) and (16), the large number of degrees of freedom of the apparatus can also provide an effective random phase to each $\alpha_n(t_0)$. In the next section, we will introduce a random phase unitary operator to describe this random phase approximation effectively, and derive the Born rule without using a partial trace.

3. Random Phase Unitary Operator and Born Rule

The group $G_\hat{O}$ in Eq. (8) is not convenient to obtain measurement outcomes, because the group contains so many evolutions that it is difficult to be described. The random phase approximation in Eq. (16) provides an effective description. Generally speaking, quantum information is exchanged under those evolutions in the group $G_\hat{O}$, and since phases are also important elements of quantum information, some phase factors can appear in those unitary evolutions of the group $G_\hat{O}$, just like the one in Eq. (15). For a particular evolution, the phases of the observed system’s states change regularly, but if there are a large number of evolutions, the phases will change in a random manner approximately. This usually occurs for a macroscopic system composing of large number of degrees of freedom, for example, a measurement apparatus, or some other general environment. Thus we can introduce an effective random phase unitary operator acting only on the observed system’s space

$$\hat{U}(\gamma) = \sum_n e^{-i\gamma_n} |n\rangle \langle n| \quad , \quad \text{all } \gamma_n \text{ are random} \quad (19)$$

with those random phases indicating an average information exchanges for the observed system. Then there will be an limit

$$G_\hat{O} \longrightarrow G_\gamma \equiv \{\hat{U}(\gamma)\} \quad (20)$$
with $G_\gamma$, a group of all the random unitary operators with the form of Eq. (19). The limit in Eq. (20) is effective, as long as the number of the elements in the group $G_\hat{O}$ is large enough. Because $[\hat{O}, \hat{U}(\gamma)] = 0$, we can rewrite the observable $\hat{O}$ as,

$$\hat{O} = \hat{U}^{-1}(\gamma)\hat{O}\hat{U}(\gamma)$$

(21)

This indicates that the observable operator $\hat{O}$ in fact does not act on a Hilbert space, but on a quotient space.

Suppose the Hilbert space of the system is

$$\mathcal{H}_S = \{|\psi\rangle, |\varphi\rangle, \cdots\}$$

(22)

then by means of the random phase unitary operator $\hat{U}(\gamma)$ or the group $G_\gamma$ in Eq. (20), we will obtain an exact quotient Hilbert space

$$\mathcal{H}_S / G_\gamma = \{\{G_\gamma |\psi\rangle\}, \{G_\gamma |\varphi\rangle\}, \cdots\}$$

(23)

still as long as the expansion coefficients of the fixed states $|\psi\rangle = \sum_n \alpha_n |n\rangle$ and $|\varphi\rangle = \sum_n \beta_n |n\rangle$ satisfy

$$|\alpha_n|^2 \neq |\beta_n|^2 \quad \text{for some } n$$

(24)

This gives an apparent coarse grained description of the original Hilbert space $\mathcal{H}_S$. Further, physical results should not depend on the random phases, thus we should construct some $\gamma$ invariant quantities in the quotient space $\mathcal{H}_S / G_\gamma$. One simple example is the density operator

$$\hat{\rho} \equiv \frac{1}{V(\gamma)} \int d\gamma \left[ \hat{U}(\gamma) |\psi\rangle \langle \psi| \hat{U}^{-1}(\gamma) \right]$$

(25)

where $V(\gamma)$ is the volume of the $\gamma$ parameter space. After some simple calculations we will have

$$\hat{\rho} = \sum_n |\alpha_n|^2 |n\rangle \langle n| \quad \text{Tr} \hat{\rho} = \sum_n |\alpha_n|^2 = 1$$

(26)

which gives the Born rule for the probability. In deriving Eq. (26), we have used

$$\int_0^{2\pi} d(\gamma_n - \gamma_m) e^{-i(\gamma_n - \gamma_m)} = 0 \quad n \neq m$$

(27)
because both $\gamma_n$ and $\gamma_m$ are random. Then by combining Eq. (5) and Eq. (26), we can obtain the definite measurement outcomes

$$\{O_n, P_n = |\alpha_n|^2\} \rightarrow \bar{O} = \sum_n O_n P_n = \text{Tr}(\hat{O}\hat{\rho})$$

(28)

It should be noted that the general space is actually $\mathcal{H}'$ in Eq. (10), or the “almost” quotient space in Eq. (12) denoted by $\mathcal{H}_t/G_{\hat{O}}$. However, the space $\mathcal{H}_t/G_{\hat{O}}$ is difficult to be described due to the complexity of the group $G_{\hat{O}}$. After a further random phase approximation, a random phase unitary operator in Eq. (19) together with the group $G_{\gamma}$ in Eq. (20) is introduced, leading to an effective quotient space $\mathcal{H}_S/G_{\gamma}$ in Eq. (23) that is easier to be described. In other words, the group $G_{\gamma}$ is a well parameterized approximation to $G_{\hat{O}}$ so that some quantities such as the density operator in Eq. (25) can be constructed effectively. These can be easily extended to the time dependent case in Eqs. (17) and (18), since a quantum measurement can be performed only on a state at some instant. Moreover, after the random phase approximation the phase information stored in each $\alpha_n$ is lost, as indicated by the randomization in Eq. (16), while the probability information $|\alpha_n|$ is still retained and can be acquired by us, as will be shown in Sec. 6.

There is a crucial difference between a Hilbert space and its corresponding (“almost”) quotient Hilbert space. In a Hilbert space, the superposition principle is applicable which roughly says that the space is closed under any linearly combinations of the states in it. However, in a (“almost”) quotient Hilbert space, its elements are not simply states, but some equivalent classes with the form of Eq. (23) or (10). The question is then whether the superposition principle is also suitable for those classes, that is, whether any linearly combinations of the equivalent classes is still some single class. To understand this question easily, we take an example about the congruence concept in number theory, for instance $1 \equiv 8 (mod 7)$, $3 = 10 (mod 7)$. That is, 1 and 8 belong to one equivalent class denoted as $\bar{1}$, while 3 and 10 belong to another class $\bar{3}$. Then we have $\bar{1} + \bar{3} = \bar{4}$, and this class addition is independent of the representative numbers.

Now consider the superposition of the equivalent classes in a (“almost”) quotient Hilbert space. For instance, we take the following three representative states from a single class

$$\frac{1}{\sqrt{2}}(e^{-i\alpha}|0\rangle + e^{-i\beta}|1\rangle)$$

13
in a quotient space of the form of Eq. (23)

\[ \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle), \quad \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \]  

(29)

Obviously, a direct addition of any two states will give states

\[ |0\rangle \text{ or } |0\rangle + \frac{i + 1}{2}|1\rangle \]

(up to some normalization constants), which belong to different classes. This indicates that the superposition principle is not suitable for equivalent classes. Certainly, for the classes of the eigenstates of some observable \( \hat{O} \), any linear combination of them is indeed some single class. This is because the representative state for any eigenstate’s class is just the eigenstate up to some irrelevant factors. However, this is just a special case, since the superposition cannot be applied further for those general classes. Then it can be concluded that the superposition principle is not suitable for the elements or equivalent classes in a (“almost”) quotient Hilbert space. This can also be seen by noting that the stability condition in the group \( G_\hat{O} \) in Eq. (8) breaks a lot of unitary symmetries. To recover the superposition principle, some evolutions noncommutative with the observable are needed to destroy the formed (“almost”) quotient space.

Therefore, when making a measurement on a quantum system, some (“almost”) quotient space can be formed according to some stability condition in Eq. (8). The superposition principle, the most important quantum property will not be suitable for the equivalent classes in the (“almost”) quotient space so that definite outcomes may be obtained. Moreover, in Sec. 5 we shall show that, this unsuitable of superposition is crucial for macroscopic systems, since their states are also the elements of some (“almost”) quotient Hilbert space. After the measurement, the quantum property of the observed system may be recovered, as long as some other evolutions noncommutative with the observable \( \hat{O} \) act on the system to destroy the formed “almost” quotient space. This is not the case for a macroscopic system whose quantum properties may still be suppressed because of its stable structure, as will be shown in Sec. 5.
4. A Two-Dimensional Example: Stern-Gerlach Experiment

As an example, we will make an analysis on the Stern-Gerlach experiment in this section, showing how the random phase unitary operator is introduced to obtain the measurement outcomes. Suppose the initial spin state of the electron is $|\phi\rangle = \alpha |\uparrow_z\rangle + \beta |\downarrow_z\rangle$, and the interaction term is given by

$$\hat{H}_{\text{int}} = -\frac{e}{2m} \hat{\sigma}_z B_z(z)$$  \hspace{1cm} (30)$$

with $B_z(z)$ the $z$-component of a non-uniform magnetic field. With this interaction, we will have a state evolution

$$|\phi\rangle \rightarrow e^{-i\hat{H}_{\text{int}}t} |\phi\rangle = \alpha e^{-i\omega t + ip_z z} |\uparrow_z\rangle + \beta e^{+i\omega t - ip_z z} |\downarrow_z\rangle$$  \hspace{1cm} (31)$$

where $\omega = |e| B_z(0) / 2m$, and

$$p_z z \simeq t F z \simeq -t |e| \frac{dB_z}{dz} (0) z$$  \hspace{1cm} (32)$$

The phase factors $e^{\pm ip_z z}$ indicate that there is momentum transfer between the electron and the magnetic field. Certainly, we can also assign $|\mp p_z\rangle_A$ as the momentum states of the apparatus or the magnetic field, and obtain the von Neumann’s proposal [6]

$$\alpha e^{-i\omega t + ip_z z} |\uparrow_z\rangle - p_z \rangle_A + \beta e^{+i\omega t - ip_z z} |\downarrow_z\rangle + p_z \rangle_A$$  \hspace{1cm} (33)$$

which gives the correlations between the electron and the measurement apparatus.

If the measurement apparatus is also a simple quantum system, the state in Eq. (33) is exact and no random phase will occur. However, this is not the case, because the magnetic field in Eq. (30) is only a classical quantity, and the full interaction should be in terms of some quantum field operators

$$\hat{H}_{\text{int}} = -\frac{e}{2m} \hat{\sigma}_z \int d^3x \hat{\psi}^\dagger (x) \hat{B}_z(x) \hat{\psi} (x)$$  \hspace{1cm} (34)$$

This means that when the electron is travelling in the magnetic field, it will always interact with the magnetic field by exchanging virtue photons so that
its trajectory will seem to be random in a smaller scale. As a result, the momentum $p_z$ in Eq. (32) is only an average quantity, because $B_z = \langle \dot{\hat{B}}_z \rangle$ has been used. Then, a more precise phase factor will be

$$\exp\left(-i\omega t + ip_z z + i\gamma\right)$$

(35)

with a random phase $\gamma$ resulting from the fluctuations due to the exchange of virtue photons.

The random phase $\gamma$ in Eq. (35) is not enough, since the phase factors for each term in Eq. (31) are not independent. This is because the evolution operator $e^{-i\hat{H}_{o}t}$ in Eq. (31) belongs to SU(2) satisfying $\text{set}(e^{-i\hat{H}_{o}t}) = 1$, since $\text{Tr } \sigma_z = 0$. For a general $U(2)$ group, there is not such a restriction. Now the states $|\mp p_z\rangle_A$ in Eq. (33) give a representation of the $U(2)$ group, so we can make an arbitrary $U(2)$ transformations $\hat{U}_A$ on them to obtain two new states. Since the states of the electron and the apparatus are entangled in Eq. (33), $\hat{U}_A$ will induce another $\hat{U}_{\text{induce}}$ transformation acting on the space of the electron. The transformation $\hat{U}_{\text{induce}}$ may be non-unitary because the electron’s new states after the transformation may be non-orthogonal. This is the so called the problem of the preferred basis or basis ambiguity in the quantum measurement, just like the case in Eq. (1). To obtain credible outcomes, we should remove this basis ambiguity coming from some general transformations involving linearly combinations of the states.

As shown below Eq. (14), there is some non-dynamical gauge that should be fixed. After that, the possible general transformations $\hat{U}_A$ will result from some dynamical evolutions. Recalling the group $G_\hat{O}$ defined in Eq. (8), the observable here is the electron’s spin $\hat{S}_z$, so we should be concerned only with those evolutions satisfying $[\hat{U}_A, \hat{S}_z] = 0$. Thus $\hat{U}_A$ can not come from an interaction with the electron, since that would violate the condition $[\hat{U}_A, \hat{S}_z] = 0$. The other source is from interactions with the rest degrees of freedom of the apparatus and a possible added environment. This means that the state in Eq. (33) should be extended to its corresponding equivalent class in an “almost” quotient space like the one in Eq. (12). For example, the representative state of the class corresponding to the first term in Eq. (33) is

$$\sum_{p_z} C_{\uparrow_z p_z} |\uparrow_z\rangle |-p_z\rangle_A |\chi_{\uparrow_z -p_z}\rangle_{\text{res}}$$

(36)
with $C_{\uparrow z, p_z}$ some transition amplitude resulting from the interactions between the electron and the apparatus, such as the phase factor in Eq. (35). Eq. (36) indicates that the states $|\uparrow p_z\rangle_A$ stand for only a small portion of the large number of degrees of freedom of the apparatus. Since the stability condition $[\hat{U}_A, \hat{S}_z] = 0$ forbids the possible source coming from interaction with the electron, another condition $[\hat{U}_A, \hat{P}_A] = 0$ is needed to forbid the possible interactions with the rest degrees of freedom. This is just the stability criterion [3] of the einselection scheme for selecting the observables of the apparatus by its environment. As will be shown in the next section, this stability condition is just one of a collection of conditions that can form a macroscopic system. Including the rest large number of degrees of freedom, a random phase approximation should be made according to the analysis below Eq. (15). This can be simply achieved by using of a random phase unitary operator

$$\hat{U}_A = e^{-i\gamma_1} |p_z\rangle \langle -p_z| + e^{-i\gamma_2} |+p_z\rangle \langle +p_z|$$

with the random phases $\gamma_1$ and $\gamma_2$ coming from the interactions with the large number of degrees of freedom of the apparatus and the possible environment. The above $\hat{U}_A$ can induce a random phase unitary transformation on the electron’s spin space

$$\hat{U}_{induce} = e^{-i\gamma_1} |\uparrow_z\rangle \langle \uparrow_z| + e^{-i\gamma_2} |\downarrow_z\rangle \langle \downarrow_z| \equiv \hat{U}_\gamma$$

just like the one of Eq. (19). By means of this random phase unitary operator and Eq. (25), we can obtain the required density operator

$$|\alpha|^2 |\uparrow_z\rangle \langle \uparrow_z| - |p_z\rangle_A \langle -p_z| + |\beta|^2 |\downarrow_z\rangle \langle \downarrow_z| + |p_z\rangle_A \langle +p_z|$$

which contains only the classical correlations between the electron’s spin states and the apparatus’s pointer states, and gives the definite measurement outcomes.

### 5. States of Macroscopic Systems

In common sense, a macroscopic systems states seem to be definite at some instant, i.e., its states behave classically. We can not perceive some superposition of states like $\alpha|\text{alive}\rangle + \beta|\text{dead}\rangle$ which says that a Schrödinger’s cat is both alive and dead. The analysis in Secs. 2 and 3 can be extended to
describe the states of those macroscopic systems. Generally speaking, the states of a macroscopic system are not in a Hilbert space, but in a (“almost”) quotient Hilbert space.

Let’s first consider a model for constructing classical bits from some qubits. Assume that there are \( N \) qubits and some interactions among them. Let the first qubit be a base with state \( |\varphi\rangle = \alpha |0\rangle + \beta |1\rangle \), while the states for other qubits are all assuming to be \( |0\rangle \) for simplicity. There is a class of primary and strong interactions (or evolutions) \( \hat{U}_{1,i} = 2, \ldots, N \) which depends also on the distance of any two qubits. Then we can further assume a decreasing relation for the strengths of these interactions

\[
\hat{U}_{1,2} \gg \hat{U}_{1,3} \gg \hat{U}_{1,4} \gg \ldots
\]

that is, the qubits are ordered in increasing distances. There are also some much weaker interactions \( \hat{U}_{i,j}, i, j = 2, \ldots, N \). This model is analogous to an atom with the first qubit as the atomic nucleus and the rest as electrons, and the couplings are the electromagnetic interactions. Consider first the first two qubits. With the interaction \( \hat{U}_{1,2} \) turned on, we have a state evolution

\[
|\varphi\rangle_1 |0\rangle_2 \xrightarrow{\hat{U}_{1,2}} \alpha e^{-i\gamma_{12}(t)} |0\rangle_1 |0\rangle_2 + \beta e^{-i\delta_{12}(t)} |1\rangle_1 |1\rangle_2
\]

which is a C-NOT gate [7] followed by some steady operator

\[
e^{-i\gamma_{12}(t)} |00\rangle_{12} (00) + e^{-i\delta_{12}(t)} |11\rangle_{12} (11)
\]

analogous to the steady states in an atom. Since the interaction \( \hat{U}_{1,2} \) is the strongest, the structure in Eq. (41) should not be destroyed by the other interactions. In other words, the rest interactions should be perturbations satisfying (ideally) \([\hat{U}_{1,2}, \hat{U}_{\text{rest}}] = 0\). Then we can obtain a group of unitary interactions

\[
G_{\hat{U}_{1,2}} \equiv \{ \hat{u}, [\hat{U}_{1,2}, \hat{U}] = 0 \}
\]

which is analogous to the one in Eq. (8). With the stability condition in Eq. (42), the rest \( \hat{U}_{i,j}, i, j = 3, \ldots, N \) can only be chosen to have the same form as \( \hat{U}_{1,2} \), i.e., a C-NOT gate followed by a corresponding steady operator; while those \( \hat{U}_{i,j}, i, j = 2, \ldots, N \) can only be some steady operators. Hence, after turning on the rest interactions, we will obtain a final state

\[
e^{-i\gamma} |0,0,\ldots,0\rangle_{1,\ldots,N} + e^{-id} |1,1,\ldots,1\rangle_{1,\ldots,N}
\]
with the random phase \( \gamma \) (in the large \( N \) limit)

\[
\gamma = \gamma_{12} + \gamma_{13} + \cdots + \gamma_{1N} + \gamma_{i,j} + \gamma_E , \quad (i, j = 2, \cdots, N)
\]

(44)
similarly for the random phase \( \delta \). The phase \( \gamma_E \) is from the surrounding environment, since the couplings with the environment must also belong to the group \( G_{\hat{U}_{1,2}} \). Otherwise no stable structure would have been formed. Just like the case in Sec. 2 and 3, the state in Eq. (43) is just the representative state of an element in an effective quotient Hilbert space, an approximation of some “almost” \( \mathcal{H}_t / G_{\hat{U}_{1,2}} \). Here, \( \mathcal{H}_t \) quotient space denoted as is the total Hilbert space of all the relevant coupled systems (\( N \) qubits) including the environment. Then by using of Eq. (25), a statistic ensemble can be obtained

\[
|\alpha|^2 |0\rangle \langle 0| + |\beta|^2 |1\rangle \langle 1|
\]

(45)
with the classical bits

\[
|0, 0, \cdots, 0\rangle_{1, \cdots, N} \equiv |0\rangle , \quad |1, 1, \cdots, 1\rangle_{1, \cdots, N} \equiv |1\rangle
\]

(46)
Although the above model is analogous to an atom, there are indeed some differences between them. In an atom, there are more energy levels than those of the qubits, and the condition in Eq. (42) may be relaxed to be some more physical one

\[
G_{\hat{U}_{stra}} \equiv \{ \hat{u}, [\hat{U}_{stra}, \hat{U}] \simeq 0 \}
\]

with \( \hat{U}_{stra} \) responsible for the structure of an atom or even a macroscopic system. This means that some perturbations can deform the already formed structure a little. Besides, an atom can not contain too many electrons, then the phases are not random enough to obtain completely classical states. This is the reason for the quantum properties of an atom. However, atoms can be combined further to form molecules, macromolecules, crystals and even all the macroscopic systems. In these processes, the involved interactions are weaker and weaker so that the already formed atomic and molecular structures are preserved. Moreover, since the number of constituents is larger and larger, the information exchanges among them are more and more complex, then the phases can be treated to be random so that the macroscopic systems’ states behave more classically. In fact, a more general (ideal) group can be proposed as

\[
G(\hat{U}_{stra}, \hat{O}_{mac}) \equiv \{ \hat{u}, [\hat{U}_{stra}, \hat{U}] = [\hat{O}_{mac}, \hat{U}] = 0 \}
\]

(47)
with \{\hat{U}_{\text{stru}}^a\} the interactions responsible for all the structures of a macroscopic system, and \{\hat{O}_{\text{mac}}^i\} standing for its macroscopic observables, such as total mass, momentum, and other macroscopic properties. The indexes \(a\) and \(i\) are used to denote the collections for those operators, which satisfy the following commutative relations

\[
[\hat{U}_{\text{stru}}^a, \hat{U}_{\text{stru}}^b] = [\hat{O}_{\text{mac}}^i, \hat{U}_{\text{stru}}^a] = [\hat{O}_{\text{mac}}^j, \hat{O}_{\text{mac}}^i] = 0
\]

(48)

This means that some quantum numbers can be assigned to denote the states of a macroscopic system. With the help of Eq. (47), the total Hilbert space can be split into equivalent classes, obtaining an “almost” quotient space in which the representative state for an equivalent class can be expressed as

\[
\sum_{a,\mu, n_i} \alpha_{\mu, n_i} \langle \mu \rangle_{\text{stru}} |n_i\rangle_{\text{mac}} \langle \chi_{\mu, n_i}\rangle_{\text{res}}
\]

(49)

with \(\mu\) and \(n_i\) the quantum numbers for the internal structures and the macroscopic observables of the macroscopic system respectively. Just like the states \(\{\chi_n\}_A\) in Eq. (13), here \(\chi_{\mu, n_i}\rangle_{\text{res}}\) for the rest degrees of freedom are also some normalized states that can provide phase factors like the one in Eq. (15). If the number of the rest degrees of freedom is large enough, a random phase approximation can also be made to give some statistical ensemble of the macroscopic system’s states, leading to their classical properties.

The stability conditions in Eq. (47) also provide a rough classification of all the interactions. The first class contains the weakest interactions satisfying all the conditions in Eq. (47). For this class, the “almost” quotient space is stable, that is, all the equivalent classes are invariant under those weakest interactions. The second class is composed of some stronger interactions that may violate some condition for the macroscopic observables, but still satisfying those for the structures. Obviously, this class of interactions can only change the macro-states of the macroscopic system, such as momentum or angular momentum as a whole. The third class is the one consisting of the strongest interactions that violate most of the conditions in Eq. (47), even those for the structures. Easily to see, the first two classes actually stand for the familiar “classical” interactions, under which the “almost” quotient space is stable or only deformed a little. While the third one can be described only by quantum theory, under which the “almost” quotient space is no longer useful for describing the macroscopic system. Certainly, there may be some
strongest interaction that violates only several conditions for the structures, for example, an X or gamma ray may destroy only some small structures of our larger bodies. In a word, the “almost” quotient space provides an effective border between the quantum and classical sides, both for the systems and the relevant interactions. If there is no “almost” quotient space in nature, the classical properties of macroscopic systems including our human beings, can not be emergent.

The above analysis implies that, the states of a macroscopic system are not simply elements of the total Hilbert space of all its constituents, but are elements of an “almost” quotient space determined by some stability conditions in Eq. (47). Besides, the structures of macroscopic systems could be formed only if the interactions were different in their strengths. This is the case in nowaday world, where there are four basic interactions whose strengths are indeed different and also depend on the distances. For instance, the strong interaction bounds quarks forming nucleons, while the weaker electromagnetic force bounds electrons and nucleons to form atoms. Since the electromagnetic force is weaker, so it cannot destroy the stable structures of the nucleons determined by the strong interaction. Further, as shown below Eq. (29), the superposition principle is not suitable for the elements of a (“almost”) quotient space, so it can be concluded that the states of macroscopic systems are more classical than quantum, as long as their stable structures are not destroyed by some other stronger interactions that violate most of the structure conditions in Eq. (47). This is the case for those macroscopic systems in ordinary temperature, while in lower temperature some macroscopic quantum phenomena can occur, such as the superconductivity and superfluidity. This can be explained by noting that the classical properties are actually caused by the large number of the rest degrees of freedom in Eq. (49). These degrees of freedom are independent from each other in ordinary temperature, while in lower temperature they behave more and more dependently so that their states’ phases become coherent and can not provide random phases. Therefore, a macroscopic system behaves classically if it contains large number of independent degrees of freedom.

There is a general picture according to the previous discussions. Since no system is absolute, thus there is also no absolute Hilbert space for an observed system in the real case, except a total one \( \mathcal{H} \), including all of the relevant coupled systems, perhaps the whole universe. If there is some local observable
\( \hat{O} \) or strongest interaction \( \hat{U}_{\text{stru}} \) singled out, some “almost” quotient space \( \mathcal{H}_f/G_\hat{O} \) or \( \mathcal{H}_f/G_{\hat{U}_{\text{stru}}} \) can be formed to describe a quantum measurement or a macroscopic system, as shown previously. The conditions in the groups \( G_\hat{O} \) and \( G_{\hat{U}_{\text{stru}}} \) imply that, the interactions violating those conditions has been neglected as long as they are weak enough not to disturb a measurement or a stable structure significantly. This is possible in the present age of our universe. Thus, a (“almost”) quotient space is only phenomenal in the sense that its formation depends greatly on the conditions of our universe.

Besides, although an observed system or a macroscopic system described by some “almost” quotient space behaves classically, the physical processes all behave in a quantum manner. With a further random phase approximation by replacing the group \( G_\hat{O} \) or \( G_{\hat{U}_{\text{stru}}} \) with some corresponding random phase group \( G_{\gamma} \), the “almost” quotient space \( \mathcal{H}_f/G_\hat{O} \) or \( \mathcal{H}_f/G_{\hat{U}_{\text{stru}}} \) can be reduced to some effective quotient space \( \mathcal{H}/G_{\gamma} \) with \( \mathcal{H} \) the relevant Hilbert space for an observed system or a macroscopic system. The quotient space \( \mathcal{H}/G_{\gamma} \) can thus be treated as the required space for the observed system or the macroscopic system, separated or decoherence from those large number of unobserved degrees of freedom effectively. Comparing with the abstract space \( \mathcal{H} \), \( \mathcal{H}/G_{\gamma} \) can only give some statistical ensemble which provides definite measurement outcomes or some classical properties of a macroscopic system. Roughly speaking, although an abstract Hilbert space \( \mathcal{H} \) is basic for describing a system, an open system can be described only by some (“almost”) quotient Hilbert space.

In an abstract Hilbert space, the superposition is applicable for its elements, but this is not the case for its corresponding (“almost”) quotient space. Then, we can answer partly the question of the Schrödinger’s cat. A superposition state of the form \( \alpha |\text{alive}\rangle + \beta |\text{dead}\rangle \) for a Schrödinger’s cat can exist only in an abstract Hilbert space, not in an “almost” quotient space for a real cat. Moreover, since the states \( |\text{alive}\rangle \) and \( |\text{dead}\rangle \) are the eigenstates of some macroscopic observable for that cat, a general state should be of the form in Eq. (49), standing for one equivalent class in an “almost” quotient space. A further analysis of the problem of the Schrödinger’s cat will be given in the next section.
6. General Quantum Measurement

A quantum measurement process is just an interaction between a quantum system and a macroscopic apparatus. As shown in Sec. 3, the definite measurement outcomes are determined by some effective quotient Hilbert space in Eq. (23). Moreover, we also show in Sec. 5 that the states of a macroscopic system are elements of some “almost” quotient Hilbert space, in which the superposition principle is not proper. It seems that the definite measurement outcomes are actually caused by the “almost” quotient Hilbert space of the macroscopic apparatus. Here we give a general analysis.

Suppose the observed system’s Hilbert space is $\mathcal{H}_S$, while the “almost” quotient Hilbert space of the apparatus is $(\mathcal{H}_T)/\hat{G}_{\hat{U}_{\text{stru}}},\hat{O}_A$, with $\mathcal{H}_T$ the total space of the apparatus and a possible added environment. We can assume that the observed system and the apparatus are completely independent initially so that $\mathcal{H}_S$ is not in $\mathcal{H}_T$. $G_{\hat{U}_{\text{stru}}},\hat{O}_A$ is a group $\{\hat{U},[\hat{U},\hat{U}_{\text{stru}}] = [\hat{U},\hat{O}_A] = 0\}$, where $\hat{U}_{\text{stru}}$ is the interaction responsible for the structure of the apparatus, and $\hat{O}_A$ is a macroscopic observable of the apparatus. Here we ignore the possible indexes of those operators for simplicity. Now, there is some interaction that can induce an evolution as

$$\mathcal{H}_S \otimes (\mathcal{H}_T/\hat{G}_{\hat{U}_{\text{stru}}},\hat{O}_A) \xrightarrow{\hat{U}_{\text{int}}} (\mathcal{H}_S \otimes \mathcal{H}_T)/\hat{G}_{\hat{O},\hat{U}_{\text{stru}}}$$

(50)

giving a measurement on the observable $\hat{O}$ of the system. The group

$$G_{\hat{O},\hat{U}_{\text{stru}}} \equiv \{\hat{U},[\hat{U},\hat{O}] = [\hat{U},\hat{U}_{\text{stru}}] = 0\}$$

includes all the interactions that don’t disturb the structure of the apparatus and the measurement outcomes, otherwise the measurement would be impossible. Eq. (50) implies that the “almost” quotient space structure of the apparatus has been transferred into the combined system via the evolution $\hat{U}_{\text{int}}$ satisfying $[\hat{U}_{\text{int}},\hat{O}_A] \neq 0$. It should be stressed that this evolution can not be a single basic unitary operator, because it also act on the apparatus’s “almost” quotient Hilbert space. Thus it must be a combination of a lot of basic evolutions constructed from the interactions between the observed system and the constituents of the apparatus and the possible environment

$$\hat{U}_{\text{int}} = \hat{U}_1\hat{U}_2\cdots \exp \left( -i \sum_n \lambda_n \hat{P}_n\hat{T}_A \right)$$

(51)
where \( \hat{P}_n \equiv |n\rangle \langle n| \) is the projector corresponding to the observable \( \hat{O} \) of the observed system. Moreover, \( \hat{U}_{\text{int}} \) belongs to the group \( G_{\hat{O},\hat{U}_{\text{stru}}} \), so \( [\hat{T}_A, \hat{U}_{\text{stru}}] = 0 \). Thus \( \hat{T}_A \) should be some macroscopic operator acting on the apparatus’s “almost” quotient Hilbert space. It can only change the macro-states for the apparatus’s macroscopic observable \( \hat{O}_A \), inducing transitions on the “almost” quotient space. For example, in the Stern-Gerlach experiment of Sec. 4, \( \hat{T}_A \approx \hat{Z}_A \), the z-component of the apparatus’s position (as a whole) [6]. It can lead to translations in momentum space obtaining the states \( |\mp p_z \rangle_A \), the momenta of the apparatus as a whole.

The evolution in Eq. (50) can also be expressed as an apparent transition of states

\[
|n\rangle_S|O_A\rangle_{\text{mac}}|\chi_{O_A}\rangle_{\text{res}} \rightarrow |n\rangle_S\left|O_A^{(n)}(t)\right\rangle_{\text{mac}}\left|\chi_{O_A^{(n)}(t)}\right\rangle_{\text{res}} \tag{52}
\]

where the apparatus’s structure part has been neglected for simplicity. The set of states \( \{|O_A^{(n)}(t)\rangle_{\text{mac}}\} \) gives a representation of the unitary evolution \( \hat{U}_{\text{int}} \). While the time parameter \( t \) indicates that those states are not exact eigenstates of the macroscopic observable \( \hat{O}_A \), since \( [\hat{O}_A, \hat{U}_{\text{int}}] \neq 0 \). After the measurement \( \hat{U}_{\text{int}} \) is absent, then we will obtain another group of stability conditions \( G_{\hat{O},\hat{U}_{\text{stru}},\hat{O}_A} \) and a corresponding “almost” quotient space in which the representative state for an equivalent class can expressed as

\[
\sum_n \alpha_n |n\rangle_S |\phi_n\rangle_T = \sum_{n,O_A^{(n)}} \alpha_n |n\rangle_S C_{O_A^{(n)}} \left|O_A^{(n)}\right\rangle_{\text{mac}}\left|\chi_{O_A^{(n)}}\right\rangle_{\text{res}} \tag{53}
\]

where the structure part is still neglected for simplicity. The state in Eq. (53) has the same form as the one in Eq. (13). Hence, we can make a further random phase approximation, by replacing the group \( G_{\hat{O},\hat{U}_{\text{stru}},\hat{O}_A} \) with a corresponding (random phase) group \( G_\gamma \) acting on either the system’s space or a larger one including the macrostates \( \{|O_A^{(n)}\rangle_{\text{mac}}\} \). By using of Eq. (25), we will then obtain the required definite measurement outcomes. For the larger space, the coefficients \( \{C_{O_A^{(n)}}\} \) can also contribute to the probabilities.

These coefficients may be from the initial state of the apparatus, or from those transition amplitudes in Eq. (52), i.e., \( \{n, O_A^{(n)}\} |\hat{U}_{\text{int}}| n, O_A^{(n)}\} \). In a real quantum measurement, the apparatus’s initial state should be prepared to be almost the same for each measurement on the prepared system’s states. Thus, the contribution from the apparatus’s initial state can be removed by
an initial random phase approximation, which prepares a definite state for the apparatus initially. While the contribution from the transitions can simply be treated as some measurement errors from the apparatus.

The expression in Eq. (53) implies that the information of the system has been diffused into the correlations with the large number of degrees of freedom of the apparatus and the possible environment. Besides, the macrostates \( \{|O_A^{(n)}\}_{mac} \) serve as the pointer states, pointing out the system’s final state after the measurement. Then whether the full information of the system’s initial state can be acquired by our human beings via some macroscopic observations on the apparatus? Macroscopic observations are actually some interactions between two macroscopic systems. Those interactions should be weak enough to satisfy most of the stability conditions for both the two macroscopic systems, then we will obtain the “classical” physical laws after some random phase approximations. For instance, our human beings, as a macroscopic system, can directly observe another macroscopic system by only weak interactions, such as by the sense of vision, touch and so on. The internal quantum properties of a macroscopic system can only be detected indirectly via some quantum tool (for example, an X ray) that interacts strongly with the macroscopic system’s constituents.

Further, the interactions between our human beings and a measurement apparatus should be so weak that even some apparatus’s macro-states, for example the pointer states, can not be disturbed by our macroscopic observations, otherwise no credible outcomes can be obtained. Those weak interactions may be expressed as

\[
\exp \left( -i \sum_{O_A} \lambda_{O_A} \hat{P}_{O_A} \hat{T}_h \right)
\]

by combining a lot of basic interactions, just like the one in Eq. (51). \( \hat{T}_h \) is a macroscopic operator acting on the “almost” quotient space of our human beings, giving our pointer states just like the case of \( \hat{T}_A \) for the apparatus. The final result is a complex correlation among the observed system, the measurement apparatus and our human beings. In particular, the system’s eigenstates, the apparatus’s pointer states and our pointer states are correlated in the same way as the expression in Eq. (2) for the einselection scheme. Therefore, our pointer states provide the perception about the state
of the combined system-apparatus, obtaining the definite outcomes with the corresponding probabilities \(\{|\alpha_n C_{O_A^{(n)}}|^2\}\). Since those weak interactions can only induce some transformations on each \(\chi_{O_A^{(n)}}\), then what about the remaining phase information stored in the phase factors of the coefficients \(\{\alpha_n C_{O_A^{(n)}}\}\)?

Quantum information can be transferred between two simple quantum systems, for example two qubits. This can be achieved by a swap operation \[7\]

\[
(\alpha|0\rangle_a + \beta|1\rangle_a)|0\rangle_b \longrightarrow |0\rangle_a (\alpha|0\rangle_b + \beta|1\rangle_b)
\]

which also means that the qubit \(b\) acquire the full information of the qubit \(a\). However, if the qubit \(b\) is replaced by a macroscopic system composing of a lot of independent degrees of freedom, there will exist various interactions other than the simple swap operation, disturbing the information transfer significantly. The result is that the information stored by one degree of freedom will be shared by almost all the independent degrees of freedom of the macroscopic system. This is analogous for a quantum measurement, as indicated by the state in Eq. (53). A definite measurement outcome contains the full quantum information, if the system’s initial state is just an eigenstate of the observable. If not, correlations must be established between the system and the apparatus, where the quantum information is stored, i.e., the information of the system’s initial state has been diffused into the apparatus. This is a general result. In order to acquire the full quantum information of the system, a recovery operation should be performed. If the apparatus was also a simple quantum system, this recover operation may be constructed easily, one simple choice is the inverse of the evolution operator \(\hat{U}_{int}\). For example, the entangled state \(\alpha |0\rangle_S |0\rangle_A + \beta |1\rangle_S |1\rangle_A\) can be obtained from an initial one \((\alpha |0\rangle_S + \beta |1\rangle_S) |0\rangle_A\) via a C-NOT gate \[7\]. Then by performing the inverse of the C-NOT gate, the information of the system can be recovered. This may also be treated as an evidence for the symmetry under \(t \rightarrow -t\) or the reversibility of any basic unitary evolution.

However, for the complex evolution in Eq. (51) that is a combination of a lot of basic evolutions, its corresponding recovery operation is too difficult to obtain, because the details of those basic evolutions are very complicated. Thus, to construct such a recovery operation, a superobserver is needed to keep track of the full details among those basic evolutions. This can also be
seen from the view of the ("almost") quotient space. Note that the elements in an "almost" quotient Hilbert space are some equivalent classes for which the superposition principle is not suitable. Thus, transitions among those classes can not be induced by a single basic unitary evolution, but only by a combination of some primary evolution together with a lot of evolutions belonging to some group $G(\cdot)$, with "\cdot" denoted as some stability conditions. Thus, transitions among those classes can not be induced by a single basic unitary evolution, but only by a combination of some primary evolution together with a lot of evolutions belonging to some group $G(\cdot)$, with "\cdot" denoted as some stability conditions. This means that the group of all unitary evolutions $\{\hat{U}\}$ can also be slit into some equivalent classes or cosets, obtaining a quotient space $\{\hat{U}\}/G(\cdot)$. This is also an abstract description for the rough classification of all the interactions given in the last section. In this sense, the evolution in Eq. (51) actually stands for some equivalent class or coset of a quotient space $\{\hat{U}\}/G(\cdot) U_{\text{stru}}, \hat{O}_A$.

Notice further that $\{\hat{U}\}/G(\cdot)$ is not a quotient group generally, may only be a semigroup, thus there is not necessary to be an inverse for each element in it. In other words, the inverse of an equivalent class or coset of evolution operators is meaningless, the symmetry under $t \rightarrow -t$ is thus broken phenomenally, leading to the irreversibility for macroscopic systems. Analogously, a recovery operation is also impossible phenomenally, that is, it can not be constructed in a practical sense.

Therefore, a macroscopic observation made by us can not be a recovery operation to acquire the full information of the system, especially the phase information. One may ask whether we can observe a quantum system directly or via some quantum tool without using a macroscopic apparatus. Analogous to the previous discussions about the quantum measurement process, a direct observation or via a quantum tool can indeed establish correlations between us and the system or the combined system-tool. That is, the information of the system or the system-tool can also be diffused into our bodies, since our human beings are also macroscopic systems. However, the brains of our human beings are not developed enough to deal with the quantum information stored in those correlations, thus a random phase approximation is necessary for us to obtain a coarse grained description of the full information. This can also be seen as follows. Although all the physical processes behave in a quantum manner, our brain as a macroscopic system is always described by an "almost" quotient space so that all the phase information is still hidden. In order to acquire the phase information, the brain’s "almost" quotient space should be destroyed by some stronger interaction. Then our brain will also be damaged so that nothing can be perceived by us. In this sense, the ran-
dom phase approximation is inevitable for us to perceive the physical world. As a result, a feeling that the information of the observed system is lost in a quantum measurement is only *phenomenal*. All the information is still stored in those correlations among the observed system, the measurement apparatus (including a possible added environment) and our human beings. *It is our undeveloped brain or its “almost” quotient space structure that prevents the full quantum information from being acquired by us.* The quantum measurement problem is thus resolved, similarly for those classical properties of our surroundings.

According to the above analysis, the problem of the Schrödinger’s cat can also be resolved in an analogous way. As a macroscopic system, the cat’s states are also elements of some “almost” quotient Hilbert space, in particular, the states \( |\text{alive}\rangle \) and \( |\text{dead}\rangle \) are the eigenstates of some macroscopic observable for that cat. The cat is coupled with a two level atom \( (E_2 > E_1) \) via a complex evolution or coupling

\[
\hat{U}_{\text{int}} \sim e^{-i\sum E_k \hat{A}_{E_k} \hat{T}_{\text{cat}}}
\]

like the one in Eq. (51). Following Eq. (50) or Eq. (52), we then have an evolution or a state transition

\[
|E_2\rangle |\text{alive}\rangle \xrightarrow{\hat{U}_{\text{int}}} e^{-i\gamma} \alpha |E_2\rangle |\text{alive}\rangle + e^{-i\delta} \beta |E_1\rangle |\text{dead}\rangle
\]

with \( \alpha \) and \( \beta \) some transition amplitudes for the atom’s states. The parts for the structures of the cat and its rest degrees of freedom have been ignored for simplicity. The random phase factors \( e^{-i\gamma} \) and \( e^{-i\delta} \) are provided by the large number of those rest degrees of freedom of the cat and a possible added environment via a random phase approximation. Hence, we will obtain a statistical ensemble with only classical correlations between the atom and the cat. In this sense, the cat can just be treated as a measurement apparatus to detect the states of the atom, but the way is somewhat immoral.

In conclusion, a quantum measurement is just a complex “unitary” interaction or evolution constructed by combining a lot of basic unitary evolutions, as given by Eq. (51). Different from those basic evolutions, this complex evolution also acts on the “almost” quotient space of the macroscopic apparatus, i.e., it is actually an element of some quotient space \( \{ \hat{U} \}/G(\cdot) \) for the unitary evolutions. Besides, this interaction establishes a correlation between the system’s measured observable \( \hat{O} \) and a macroscopic one \( \hat{O}_A \) of the apparatus.
When the apparatus is observed by our human beings, the involved interactions lead to a more complex correlation among the observed system, the measurement apparatus and our human beings, in particular among the system’s eigenstates, the pointer states of the apparatus and our human beings. A random phase approximation is needed because the interactions involved in the observation can not recover the full quantum information, especially the phase information. As a result, only definite outcomes with the corresponding probabilities can be acquired by us. This means that there is an intrinsical limit for our human beings not to acquire the full (phase)quantum information, due to our brain’s stable “almost” quotient space structure. This is consistent with the reason for introducing decoherence, the inability to keep track of the large number of (independent) degrees of freedom of the relevant environment.

7. Conclusions

It’s worth to make a comparison between our random phase approach and the environment-induced superselection or einselection scheme. The same point is that both of them use the concept of decoherence, i.e., the observed system is open to its surroundings. But the methods to induce a decoherence are different, a partial trace for the einselection scheme while a random phase approximation for our approach. The condition for a partial trace seems to be more strict indicated by the orthogonal condition $\langle E_1 | E_2 \rangle \approx 0$ for the environment’s states. This makes the einselection scheme concern mainly with some “local” details of the coupled systems, for example the possible forms of the relevant interactions and states, by proposing some models to describe the decoherence [1, 3, 8-10]. While our approach provides a “global” view about some general properties of an open system, by forming an “almost” quotient Hilbert space according to some stability conditions. When forming that “almost” quotient space, a lot of unitary symmetries are broken so that the superposition principle is no longer proper generally. As a result, definite measurement outcomes may be obtained via a further random phase approximation.

In Sec. 1, we have listed five problems of the einselection scheme, here are the resolutions of our random phase approach. The first one is about deriving the Born rule. A partial trace applied in the einselection scheme seems
to use circular arguments. While in our approach after a random phase approximation, an integration over the random phases’ space is performed to obtain the Born rule, as given by Eq. (25). The second problem is about the role played by the environment and the stability criterion. In the einselection scheme, the environment seems to play a dominate role, by selecting or determining the apparatus’s pointer states with the help of the stability criterion. In our approach, an analogous stability condition for the observed system is proposed to construct a subgroup of evolutions that serves as a necessary prerequisite to make a quantum measurement, as shown in Sec. 2. When the group is acted on the observed system, an “almost” quotient Hilbert space can be formed. In a quantum measurement process, the apparatus’s pointer states are determined mainly by the interaction between the system and the apparatus, but are stabilized by some more stability conditions that may involve a possible added environment, as shown in Sec. 6.

The third problem is about the orthogonal condition \( E_1 \parallel E_2 \approx 0 \) for the environment’s states, which may not be satisfied generally. In our approach, the random phase approximation is effective as long as the number of the relevant environment’s (independent) degrees of freedom is large enough. Certainly, when the environment is also a simple quantum system or lives in lower temperature, random phase approximation is no longer proper since no “almost” quotient space can be formed. In this case, no decoherence can occur except that an extra observation is made by us, involving a macroscopic apparatus. The fourth problem is about the large number of unobserved degrees of freedom of a macroscopic apparatus, which may also induce a decoherence without an extra environment. In our approach, these unobserved degrees of freedom help to form the (“almost”) quotient space of the macroscopic apparatus, and also provide random phases to lead to the apparatus’s classical properties, as shown in Sec. 5. The last problem is about demonstrating the reason of the decoherence. The reason for introducing decoherence is due to our inability to keep track of an environment’s large number of degrees of freedom. However, the partial trace can be used as a more general operation without involving this reason. In our approach, this reason is demonstrated apparently by the condition of the random phase approximation, i.e., in the limit of large number of independent degrees of freedom, as indicated by Eq. (16). This implies that decoherence is not a physical process, but only an approximative method for us to describe or observe the world.
In conclusion, a quantum measurement is a complex “unitary” interaction, with the complexities coming from the large number of independent degrees of freedom of the macroscopic apparatus and a possible added environment. The classical properties of the macroscopic apparatus are mainly caused by its large number of degrees of freedom, and can be described by the emergence of an “almost” quotient Hilbert space phenomenally, determined by some necessary stability conditions. After the measurement, the combined system-apparatus can still be described by another “almost” quotient Hilbert space with some classical properties. Moreover, our macroscopic observations on the combined system-apparatus can not recover the full quantum information, especially the phase information. A random phase approximation is thus needed to give only a coarse grained description of the full quantum information. As a result, only the information stored in the remaining classical correlations among the observed system, the apparatus and our human beings, can be acquired by us. This gives the definite measurement outcomes perceived by us with the corresponding probabilities.

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References

