

# A simple derivation of Born's rule with and without Gleason's theorem.

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## Abstract

We present a derivation of Born's rule and unitary transforms in Quantum Mechanics, from a simple set of axioms built upon a physical phenomenology of quantization. Combined to Gleason's theorem, this approach naturally leads to the usual quantum formalism, within a new conceptual framework that is discussed heuristically in details. The structure of Quantum Mechanics, from its probabilistic nature to its mathematical expression, appears as a result of the interplay between the quantized number of "modalities" accessible to a quantum system, and the continuum of "contexts" that are required to define these modalities.

## 1. Introduction

Deriving Born's rule, rather than postulating it as it is done in standard textbooks, has been envisioned since the early times of Quantum Mechanics (QM) [1]. A major asset in this direction is Gleason's theorem [2], whose critical importance for the foundations of QM has been recognized since it was published in 1957. The theorem is simple to state (see below), but difficult to demonstrate, and a nice presentation is provided in [3].

The main attempts to use Gleason's theorem for deriving Born's rule, and then the whole quantum formalism, have been done in the framework of formal quantum logic [4]. However, such approaches were not considered very appealing by physicists, and though Gleason's theorem essentially gives the correct quantum probability law, it was often said that it provides no physical insight into why the result should be regarded as probabilities. According to [5], it is even considered as a motivation to seek a more physically transparent derivation of Born's rule. This is partly because the hypothesis of Gleason's theorem do not fit easily in the usual "wave function" approach of QM, and in particular within the superposition principle, which is usually put forward as the very first postulate when introducing quantum mechanics.

In this paper, we will introduce new axioms for QM [6-14], starting with three physical axioms defining quantum rules, without any mathematical formalism. When completed by a fourth mathematical axiom, it will turn out that the four together correspond to the hypothesis of Gleason's theorem, leading straightforwardly to Born's rule. Before stating this fourth axiom,

we will introduce it heuristically, by using the three physical ones, completed by a set of physical assumptions. So let us start with the following set of physically motivated axioms, which have been introduced and discussed in [15-17]:

- **Axiom 1** (quantum ontology): Given a physical system, a *modality* is defined as the values of a complete set of physical quantities that can be predicted with certainty and measured repeatedly on this system. The complete set of physical quantities is called a “context”, and the modality is attributed jointly to the system and the context.
- **Axiom 2** (quantization): For a given context, that is a given “knob settings” of the measurement apparatus, there exist  $N$  distinguishable modalities  $\{u_i\}$ , that are mutually exclusive : if one is true, or verified, the other ones are wrong, or not verified. The value of  $N$ , called the dimension, is a characteristic property of a given quantum system.
- (changing contexts): The different contexts relative to a given quantum system are related between themselves by (classical) transformations  $g$  that have the structure of a continuous group  $\mathcal{G}$ .

For the sake of clarity, we note that, within the usual QM formalism (not used so far), a modality and a context correspond respectively to a pure quantum state, and to a complete set of commuting observables. The axioms are formulated for a finite  $N$ , but this restriction will be lifted below. These axioms, under the acronym “CSM”, meaning Context, System, Modality, have been discussed in [17], both physically and philosophically, and we will not reproduce this discussion here. We will rather consider the following question: it is postulated in Axiom 2 that there are  $N$  mutually exclusive modalities associated to each given context, but there are many more modalities, corresponding to all possible contexts. These modalities are generally not mutually exclusive, but are **incompatible**: it means that if one is true, one cannot tell whether the other one is true or wrong. Then, how to relate between themselves all these modalities?

A first crucial result already established in [17] is that this connection can only be a probabilistic one, otherwise the axioms would be violated; the argument is as follows. Let us consider a single system, two different contexts  $C_u$  and  $C_v$ , and the associated modalities  $u_i$  and  $v_j$ , where  $i$  and  $j$  go

from 1 to  $n$ . The quantization principle (Axiom 2) forbids to gather all the modalities  $u_i$  and  $v_j$  in a single set of  $2N$  mutually exclusive modalities, since their number is bounded to  $N$ . Therefore the only relevant question to be answered by the theory is: If the initial modality is  $u_i$  in context  $C_u$ , what is the *conditional probability* for obtaining modality  $v_j$  when the context is changed from  $C_u$  to  $C_v$ ? We emphasize that this probabilistic description is the unavoidable consequence of the impossibility to define a unique context making all modalities mutually exclusive, as it would be done in classical physics. It appears therefore as a joint consequence of the above Axioms 1 and 2, i.e., that modalities are quantized, and require a context to be defined.

Now, according to Axiom 3, changing the context results from changing the measurement apparatus at the macroscopic level, that is, “turning knobs”. A typical example is changing the orientations of a Stern-Gerlach magnet. These context transformations have the mathematical structure of a continuous group, denoted  $\mathcal{G}$ : the combination of several transformations is associative and gives a new transformation, there is a neutral element (the identity), and each transformation has an inverse. Generally this group is not commutative : for instance, the three-dimensional rotations associated with the orientations of a Stern-Gerlach magnet do not commute. For a given context, there is a given set of  $N$  mutually exclusive modalities, denoted  $\{u_i\}$ . By changing the context, one obtains  $N$  other mutually exclusive modalities, denoted  $\{v_j\}$ , and one needs to build up a mathematical formalism, able to provide the probability that a given initial modality  $\{u_i\}$  ends up in a new modality  $\{v_j\}$ .

The standard approach at this point is to postulate that each modality  $u_i$  is associated with a vector  $|u_i\rangle$  in a  $N$ -dimensional Hilbert space, and that the set of  $N$  mutually exclusive modalities in a given context is associated to a set of  $N$  orthonormal vectors. Rather than vectors  $|u_i\rangle$  and  $|v_j\rangle$ , one can equivalently use rank-one projectors  $P_u$  and  $P_v$ , and Born’s rule giving the conditional probability  $p(v_j|u_i)$  can be written as

$$p(v_j|u_i) = \text{Tr}(P_{u_i}P_{v_j}) \quad (1)$$

As we will show below, after postulating that each modality is associated to an Hermitian projector acting in a suitable Hilbert space, there is actually no need to postulate also Born’s rule: it follows immediately as a consequence of Gleason’s theorem, and the transformation of projectors associated with

a change of context must be unitary. Before doing that, we shall first justify heuristically why each modality should be associated to a projector. This will be done in Section 2, then we will come back to Gleason's theorem in Section 3, and finally discuss some consequences of our approach in Section 4.

## 2. Heuristics without Gleason's theorem

The main goal of this heuristics is to give a justification for Axiom 4 (given in Section 3 below), telling that each modality is associated to an Hermitian projector acting in a suitable Hilbert space. For this purpose, we will start from Axioms 1-3 only, and introduce a set of assumptions to construct a consistent probability theory, by imposing some requirements on what it should describe. This will lead us to associate modalities with projectors in an Hilbert space, and to get Born's rule and unitary transforms on the way. The more formal proofs will be given in Section 3 by using Axiom 4.

### The general probability matrix

Since the  $\{u_i\}$  and  $\{v_j\}$  are by definition non-exclusive modalities, one has to introduce the probabilities of finding the particular modality  $v_j$  (in the new context), when one starts in modality  $u_i$  (in the old context). There are  $N^2$  such probabilities, that can be arranged in a matrix  $\Pi_{v|u} = (p(v_j|u_i))$ , containing all probabilities connecting the  $N$  modalities in each context  $\{u_i\}$  and  $\{v_j\}$ . Since one has obviously  $0 \leq p(v_j|u_i) \leq 1$  and  $\sum_j p(v_j|u_i) = 1$ , the matrix  $\Pi_{v|u}$  is said to be a *stochastic* matrix.

For clarity, let us emphasize the interpretation of the conditional probability notation: in agreement with the definition of modalities as certainties, the meaning of  $p(v_j|u_i)$  is that "if we start (with certainty) from modality  $u_i$  in the old context, then the probability to get modality  $v_j$  in the new context is  $p(v_j|u_i)$ ". The matrix of all  $p(v_j|u_i)$  provides the starting point for our heuristic approach, by which theoretical predictions are connected to experiments. For  $N = 3$ , one will have for instance

$$\Pi_{v|u} = \begin{pmatrix} p(v_1|u_1) & p(v_2|u_1) & p(v_3|u_1) = 1 - p(v_1|u_1) - p(v_2|u_1) \\ p(v_1|u_2) & p(v_2|u_2) & p(v_3|u_2) = 1 - p(v_1|u_2) - p(v_2|u_2) \\ p(v_1|u_3) & p(v_2|u_3) & p(v_3|u_3) = 1 - p(v_1|u_3) - p(v_2|u_3) \end{pmatrix}$$

As we will see below,  $N \geq 3$  is required because some crucial properties of  $\Pi_{v|u}$  do not show up for  $N = 2$ .

Let us also define a “return” probability matrix  $\Pi_{u|v}$ , by exchanging the roles of the initial and final contexts. The matrix  $\Pi_{u|v}$  has the same properties as  $\Pi_{v|u}$ , but these two matrices are *a priori* unrelated, whereas it is well known that in standard QM, they are transpose of each other. In the following, we will introduce simple assumptions which will constraint these matrices to being *unistochastic*, i.e., that their coefficients are the square moduli of the coefficients of a unitary matrix [18]; and then, to being transpose of each other.

## A mathematical identity

In order to manipulate the  $\Pi_{v|u}$  and  $\Pi_{u|v}$  matrices, it is convenient to introduce orthogonal ( $N \times N$ ) projectors  $P_i$ , that are zero everywhere, except for the  $i^{th}$  term on the diagonal that is equal to 1. These projectors verify the relation  $P_i P_j = P_i \delta_{ij}$ . A useful operation is then to extract the particular probability  $p(v_j|u_i)$  from  $\Pi_{v|u}$ , or  $p(u_i|v_j)$  from  $\Pi_{u|v}$ , and one has the following identities:

$$p(v_j|u_i) = \text{Tr}(P_j \Sigma_{v|u}^\dagger P_i \Sigma_{v|u}) = \text{Tr}(P_i \Sigma_{v|u} P_j \Sigma_{v|u}^\dagger) \quad (2)$$

$$p(u_i|v_j) = \text{Tr}(P_i \Sigma_{u|v}^\dagger P_j \Sigma_{u|v}) = \text{Tr}(P_j \Sigma_{u|v} P_i \Sigma_{u|v}^\dagger) \quad (3)$$

where  $\text{Tr}$  is the Trace,  $\dagger$  is the Hermitian conjugate, and

$$\Sigma_{v|u} = [e^i \phi_{v_j|u_i} \sqrt{p_{v_j|u_i}}] , \quad \Sigma_{u|v} = [e^i \phi_{u_i|v_j} \sqrt{p_{u_i|v_j}}] \quad (4)$$

are  $N \times N$  matrices formed by square roots of the probabilities, and by arbitrary phase factors which are introduced for the sake of generality, and don't play any role at that stage. We emphasize that the equations above are only mathematical identities, and don't tell more than what is already contained in the definition of the matrices  $\Pi_{v|u}$  and  $\Pi_{u|v}$ . A useful marginal case is the situation where the context is not changed, so  $u \equiv v$ , and

$$p(u_j|u_i) = \text{Tr}(P_j P_i) = \delta_{ij} \quad (5)$$

where  $p(u_j|u_i) = \delta_{ij}$  is obviously consistent with mutually exclusive modalities within a given context.

From Eqs. 2, 3 the elements  $p_{ji}$  of a general stochastic matrix  $\Pi$  can be written as (the subscripts  $u|v$  or  $v|u$  are omitted for simplicity):

$$p_{ji} = \text{Tr}(P_i \Sigma P_j \Sigma^\dagger) \quad (6)$$

Now, according to the singular values theorem, there must exist two unitary matrices  $U$  and  $V$ , and a real diagonal matrix  $R$ , such that

$$\Sigma = URV^\dagger, \Sigma^\dagger = VRU^\dagger \quad (7)$$

where the diagonal values of  $R$  are the square roots of the (real) eigenvalues of  $\Sigma \Sigma^\dagger$ , equal to those of  $\Sigma^\dagger \Sigma$ , and are called the singular values of  $\Sigma$  [19]. The matrix  $\Sigma$  is unitary iff  $R$  is the identity matrix  $\hat{1}$ . We note that the value of  $\text{Tr}(R^2)$  is the sum of the square moduli of all the coefficients of  $\Sigma$ , and is therefore equal to  $N$ . For a generic stochastic matrix  $\Pi$ ,  $\Sigma \Sigma^\dagger$  has diagonal coefficients equal to 1, but is not diagonal, whereas  $R^2$  is diagonal, and its  $N$  coefficients are real, positive, and sum to  $N$ , but are not necessarily equal to one.

Using Eqs. (6, 7),  $p_{ji}$  can now be written:

$$p_{ji} = \text{Tr}(P_i URV^\dagger P_j VRU^\dagger) = \text{Tr}((U^\dagger P_i U) R (V^\dagger P_j V) R) \quad (8)$$

This equation is again a mathematical identity, on which we shall now impose physical constraints. In the section below we will consider  $\Sigma_{v|u}$ , but obviously the same arguments are also valid for  $\Sigma_{u|v}$ .

## Physical constraints on the probability matrix

Given Axioms 1 and 2, our main physical argument is that the probability  $p_{v_j|u_i}$  should only depend on the particular modalities  $u_i$  and  $v_j$  being considered, and not on the whole contexts in which they are embedded. This important property of “non-contextuality” for modalities [20] is related to Gleason’s theorem, and it will appear again in Section 3. It tells that the same modality can pertain to different contexts, and therefore can be defined (in particular, mathematically) independently of other modalities in a given context. This (quantum) non-contextuality is fully compatible with contextual objectivity [15-17]: the latter states that a modality needs a context to be defined, whereas the former tells that the same modality can be defined

in several contexts.

In order to fulfill this condition, the decomposition of Eq. (8) suggests that it might be possible to separate two parts (within parenthesis) associated with the two specific modalities  $u_i$  and  $v_j$ . However, if the singular values of the matrix  $\Sigma_{v|u}$  are not equal to 1, the matrix  $R \neq \hat{1}$  will impose a context-dependent structure on the whole sets of modalities  $\{u_i\}$  and  $\{v_j\}$ . There is nevertheless a way to warrant that  $R$  does not depend on  $\Sigma_{v|u}$ , still satisfying the constraints spelled out above: it is to impose that  $R = \hat{1}$ . Therefore, in order to have the probability depending on separate mathematical objects associated with each modality, we will posit the basic assumption:

- Assumption 1: In order to ensure that  $p_{v_j|u_i}$  depends only on the two particular modalities being considered, the  $N$  singular values of  $\Sigma_{v|u}$  must be all equal together, and thus are all equal to one.

Then as said above  $\Sigma_{v|u}$  will be a unitary matrix  $UV^\dagger$ , but one may wonder whether orthogonal (real) matrices might be enough. In order to justify that the full unitary set is required, we shall use a second assumption:

- Assumption 2: Since the change of contexts corresponds to a continuous group (Axiom 3), the set of matrices  $\Sigma_{v|u}$  must be connected in a topological sense, and must contain the identity matrix.

Then it is known that the set of orthogonal matrices is topologically disconnected in two parts with determinant +1 and -1, which contradicts the above assumption. For instance, permutation matrices are not connected to the identity, whereas they correspond simply to “relabelling” the modalities, i.e., to a trivial change of context. On the other hand, all (complex) unitary matrices are connected to the identity, and do agree with Assumption 2 (for other arguments see refs. [21-23]).

## Unitary matrices and Born’s formula

We are thus lead to the conclusion that  $\Sigma_{v|u}$  must be a unitary matrix  $S_{v|u}$ , with  $S_{v|u}^\dagger = S_{v|u}^{-1}$ . Then Eqs. (2) for picking up a particular probability become:

$$p_{v_j|u_i} = \text{Tr}(P_j \cdot S_{v|u}^\dagger \cdot P_i \cdot S_{v|u}) = \text{Tr}(P_i \cdot S_{v|u} \cdot P_j \cdot S_{v|u}^\dagger) \quad (9)$$



which shows that the matrix  $\Pi_{v|u}$  must be *unistochastic*, i.e., made by the square modulus of the coefficients of a unitary matrix. Such matrices are also *bistochastic*, i.e., their lines and rows sum to 1 [24]. Then we can define

$$P'_i = S_{v|u}^\dagger \cdot P_i \cdot S_{v|u}, \quad P''_j = S_{v|u} \cdot P_j \cdot S_{v|u}^\dagger \quad (10)$$

It is clear that these operators are all Hermitian projectors, i.e., one has  $P^\dagger = P$  and  $P^2 = P$  for each of them, and also that all sets  $\{P'_i\}$  and  $\{P''_j\}$  have the same orthogonality properties as the initial set of projectors  $\{P_i\}$ , given by Eq. (5). One can thus rewrite Eq. (2) as:

$$p_{v_j|u_i} = \text{Tr}(P_i P'_i) = \text{Tr}(P_i P''_j) \quad (11)$$

This is just Born's formula (Eq. 1), which is obtained here heuristically, rather than from a postulate.

Finally, the obvious next step is to associate projectors with modalities in each context, and for the matrix  $\Pi_{v|u}$  it can be done in two consistent ways as seen above:

$$\begin{aligned} \text{old context}\{u_i\} &\rightarrow \text{new context}\{v_j\} \\ P'_i &= S_{v|u}^\dagger \cdot P_i \cdot S_{v|u} \rightarrow P_j \\ P_i &\rightarrow P''_j = S_{v|u} \cdot P_j \cdot S_{v|u}^\dagger \end{aligned} \quad (12)$$

One can now come back to the matrix  $\Pi_{u|v}$ , for which the same reasoning is valid, and leads to a unitary matrix  $S_{u|v}$ . By reverting the contexts one has thus:

$$\begin{aligned} \text{old context}\{v_j\} &\rightarrow \text{new context}\{u_i\} \\ Q''_j &= S_{u|v}^\dagger \cdot P_j \cdot S_{u|v} \rightarrow P_i \\ P_j &\rightarrow Q'_i = S_{u|v} \cdot P_i \cdot S_{u|v}^\dagger \end{aligned} \quad (13)$$

But since projectors are now associated with modalities, they should be the same for a given modality in a given context, i.e., one should have  $P''_j = Q''_j$  and  $P'_i = Q'_i$ . This is obtained if  $S_{u|v}$  is the inverse of  $S_{v|u}$ , leading to a third assumption:

- Assumption 3: In order to associate projectors with modalities in a consistent way, the matrices  $\Pi_{u|v}$  and  $\Pi_{v|u}$  must be related by  $S_{u|v} = S_{v|u}^\dagger = S_{v|u}^{-1}$ , and thus  $\Pi_{u|v} = \Pi_{v|u}^t$

Then the various point of views represented in the relations (13, 14) are all consistent and give the same values for the probabilities, because each  $S_{v|u}$  can be associated to an element of the group of context transformations  $\mathcal{G}$ , and its inverse is  $S_{u|v} = S_{v|u}^{-1} = S_{v|u}^\dagger$ . For the general consistency of the approach, this set of matrices gives a  $N \times N$  (projective) representation of the group of context transformations; this is fully consistent with the well known Wigner theorem [25]. This continuous unitary evolution will be essential to describe the evolution of the system (translation in time), and it is also related to Theorem 2 in Sec. 3 below.

The identification of the matrix  $\Sigma_{v|u}$  as being a unitary matrix  $S_{v|u}$ , and the association of projectors to modalities, are the results we were looking for; in the next section they will be restated as Axiom 4 in our framework. In the above heuristic calculation, valid in the finite-dimensional case, they appear to be a joint consequence of the three assumptions made above, and of the mathematical identity given by Eqs. (2, 3). By starting from Axiom 4, the Trace formula used in this identity will turn out to be the only possible choice.

We also obtained Born's formula, apparently avoiding the heavy machinery of Gleason's theorem, because we use the tools of linear algebra applied to real or complex  $N \times N$  matrices, where all the required mathematical properties are already embedded. In order to obtain a full mathematical proof, we will now formally state Axiom 4, and deduce Born's rule in the general case.

### 3. Born's rule from Gleason's theorem

Here we add explicitly a fourth axiom, which associates modalities with projectors in a Hilbert space. Then we will demonstrate two theorems, which are respectively Born's rule, and the unitary evolution of projectors. The axiom and the theorems are as follows.

- **Axiom 4:** For a system with dimension  $N$ , each modality is bijectively associated with a  $N \times N$  Hermitian rank-one projector  $P_i$  ( $P_i^\dagger = P_i^2 = P_i$ ). Each set of  $N$  modalities within a given context is associated to a set of  $N$  such projectors, verifying  $P_i P_j = P_i \delta_{ij}$ , and  $\sum_i P_i = \hat{1}$ . The same projector (and therefore the same modality) may be part of different contexts.

- **Theorem 1 (Born's formula):** If the system is known to be in the modality  $u_i$  from the set  $\{u_i\}$ , the probability that it is found in modality  $v_j$  from the set  $\{v_j\}$  corresponding to another context obtained by the context transformation  $g_{v|u}$  is:

$$p_{v_j|u_i} = \text{Tr}(P_i \cdot P'_j) \quad (14)$$

where  $P_i$  and  $P'_j$  are respectively associated to the modalities  $u_i$  and  $v_j$ .

- **Theorem 2 (unitary transforms):** The different sets of projectors corresponding to different contexts are related by unitary transformations  $S_{v|u}$ , i.e., one has  $P'_j = S_{v|u} \cdot P_j \cdot S_{v|u}^\dagger$ .

*Proof.* Let us first remind Gleason's theorem [2, 3] :

*Let  $f(P_i)$  be a function of rank-one projectors  $P_i$  in a real or complex Hilbert space with a dimension larger than 2, to the interval  $[0,1]$  of real numbers. Let assume that  $\sum_i f(P_i) = 1$  for any set  $\{P_i\}$  of mutually orthogonal projectors ( $P_i P_j = P_i \delta_{ij}$ ) verifying  $\sum_i P_i = Id$ . Then there is a unique positive Hermitian operator  $\rho$  with unit trace so that  $f(P_i) = \text{Tr}(\rho P_i)$  for all  $P_i$ .*

According to Axiom 4, each modality is bijectively associated to an Hermitian rank-one projector  $P_i$ , and any context is associated to a set of  $N$  mutually orthogonal projectors  $\{P_i\}$  verifying  $\sum_i P_i = Id$ . Let us start from a context  $\{P_i\}$ , and go to another context  $\{P'_j\}$ , which may actually be the same as  $\{P_i\}$ . Since one has necessarily  $\sum_j p(P'_j|P_i) = 1$  for any set  $\{P'_j\}$ , there exists a unique  $\rho$  such that  $p(P'_j|P_i) = \text{Tr}(P'_j \rho)$ .

In addition, one may choose  $P'_j = P_i$ , and then (since  $\rho$  is unique)  $p(P_i|P_i) = \text{Tr}(P_i \rho) = 1$ . This is possible only if  $\rho = P_i$ , and we obtain the expected Born's formula  $p(P'_j|P_i) = \text{Tr}(P'_j P_i)$ . This proves Theorem 1.

In addition, since  $\{P_i\}$  and  $\{P'_j\}$  are sets of projectors onto two orthonormal basis (Axiom 4), there is a unitary transform  $S_{j|i}$  such that  $P'_j = S_{j|i} \cdot P_j \cdot S_{j|i}^\dagger$ , up to some relabelling of the basis. This proves Theorem 2.

Let us note that the main hypothesis for Gleason's theorem, i.e., that probabilities sum to one for any set of mutually orthogonal projectors summing

to identity, is a joint consequence of Axiom 2, i.e. that there are  $N$  mutually exclusive modalities in each given context, and Axiom 4, i.e., that mutually orthogonal projectors are associated to these modalities. This remark allows us to lift the restriction on a finite value of  $N$ : since Gleason's theorem is valid in any dimension, Axioms 2 and 4 can also be considered valid for any  $N$  [26]. This means also that the (classical) additivity of probabilities can be used within a given context [27].

Another more implicit hypothesis is that  $f(P_i)$  depends only on  $P_i$ , and not on other (orthogonal)  $P_{j \neq i}$  within the given set in  $\sum_i P_i = Id$ ; this property is usually called “non-contextuality” [20], and we already introduced it as Assumption 1. It means that, given an initial modality, the conditional probability depends on the particular outcome modality considered, and not on other modalities within a given outcome context. Though this hypothesis may be considered very strong [28], it fits perfectly with our “objective” definition of modalities [15-17]: though a modality needs a context to be observed, the same modality may appear in different contexts, always associated with the same projector [29]. Therefore the **physical** Axioms 1-3, complemented by the mathematical Axiom 4, do allow us to deduce Born's rule from Gleason's theorem.

## 4. Discussion

Since we have now reached the starting point of most QM textbooks [30], it should be clear that the standard structure of QM can be obtained from the above axioms [31]. In particular, one can associate the  $N$  orthogonal projectors  $\{P_i\}$  to the  $N$  orthonormal vectors which are eigenstates of these projectors up to a phase factor, i.e., to rays in the Hilbert space. Similarly, the expected probability law for the measurement results  $\{a_i\}$  will be obtained by writing any physical quantity  $A$  as an operator  $\hat{A} = \sum_i a_i P_i$ , this is the usual spectral theorem.

We emphasize that we do not need any additional “measurement postulate”, since measurement is already included in Axiom 1, i.e., in the very definition of a modality (see detailed discussions in [15-17]). Quantum superpositions are certainly there as usual, but they are not spooky “dead-and-alive” concepts: they are rather the manifestation of a modality (i.e., a certainty) in

another context. Entanglement is also present as linear superpositions of tensor product states, corresponding to modalities in a “joint” context. In a two-particle Bell-EPR experiment [32, 33], the entangled modality is defined in a joint context (e.g., a singlet state for two spins), and it is incompatible with a separable modality corresponding to separate measurements. When a measurement is done on one side for one particle only, there is no influence or action at a distance, but the system (still not measured on the other side) may be quite far from the new context resulting from the partial measurement. Since a modality requires both a context and a system, one sees that it embeds non-local features, corresponding to quantum non-locality, but fully compatible both with relativistic causality and with physical realism [17].

The view about the “classical vs quantum” dilemma that emerges from our approach has been discussed in details in [17]. It does agree with physical realism, given that classical objectivity has been replaced by contextual objectivity [15, 16], as expressed by Axiom 1. This Axiom takes from EPR their definition of “elements of physical reality” [34] based on full predictability and reproducibility, and from Bohr the idea that such a physical reality must include “the very conditions which define the possible types of predictions regarding the future behavior of the system” [35], i.e., the context. Therefore physical reality does not belong any more to the system alone, but pertains jointly to the Context, System, and Modality (CSM). This approach allows one to distinguish clearly between the modality, which is basically a real physical phenomenon, or a physical event in the sense of probability theory, and the projector, which is a mathematical tool for calculating non-classical probabilities. This point of view also provides novel answers to questions about the “reality of the wave-function”.

To conclude, let us emphasize that we discussed a very idealized version of QM, based on pure states and orthogonal measurements. Nevertheless, this idealized version does provide the basic quantum framework, and connects the experimental definition of a physical quantity and the measurement results in a consistent way, both physically and philosophically [17]. Adding more refined tools such as density matrices, imperfect measurements, POVM, open systems, decoherence, is of great practical interest and use, but this will not “soften” the basic ontology of the theory, as it is presented here. The present work, deeply rooted in ontology, is thus complementary to many recent related proposals [6-14].

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Then define another unitary matrix  $U$  such that  $UR = \Sigma V$ , and  $RU^\dagger = V^\dagger \Sigma^\dagger$ , so that  $U^\dagger \Sigma^\dagger \Sigma U = R^2$ . The one gets  $\Sigma = URV^\dagger$  as expected.

- [20 ] This definition of non-contextuality is used in articles dealing with Gleason’s theorem [3], and it is sometimes called “measurement non-contextuality” [14]. In that sense QM is non-contextual, without any contradiction with the Kochen-Specker theorem.
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- [24 ] *Orthostochastic* and *unistochastic* matrices are obtained by taking the square moduli of the coefficients of respectively an orthogonal or a unitary matrix [18]. For  $N = 2$ , all bistochastic matrices are also ortho- and unistochastic; for  $N \geq 3$ , the set of unistochastic matrices is larger than the orthostochastic set, but smaller than the bistochastic set. For instance, the simple matrix

$$\frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

is a well known example of a bistochastic matrix, which is neither orthostochastic, nor unistochastic; therefore it is not an acceptable probability matrix  $\Pi_{u|v}$ .

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- [29 ] Consider e.g. a system of two spin  $1/2$  particles, and define  $\vec{S} = \vec{S}_1 + \vec{S}_2$ . Using standard notations for coupled and uncoupled basis, the  $|m_1 = 1/2, m_2 = 1/2\rangle$  modality in the context  $\{S_{z1}, S_{z2}\}$  is the same as the  $|S = 1, m_S = 1\rangle$  modality in the context  $\{\vec{S}^2, S_z\}$ , though other modalities in the same two contexts are different.
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- [31 ] The postulate on time evolution is not spelled out, but it enters in the same framework, by including translations in time in the group  $\mathcal{G}$ . For instance, if  $\mathcal{G}$  is the Galileo group, standard non-relativistic QM can be recovered, including Schrödinger’s equation [25].
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