Quantum-Bayesian Coherence: The No-Nonsense Version

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Abstract

In the Quantum-Bayesian interpretation of quantum theory (or QBism), the Born Rule cannot be interpreted as a rule for setting measurement-outcome probabilities from an objective quantum state. But if not, what is the role of the rule? In this paper, we argue that it should be seen as an empirical addition to Bayesian reasoning itself. Particularly, we show how to view the Born Rule as a normative rule in addition to usual Dutch-book coherence. It is a rule that takes into account how one should assign probabilities to the consequences of various intended measurements on a physical system, but explicitly in terms of prior probabilities for and conditional probabilities consequent upon the imagined outcomes of a special *counterfactual* reference measurement. This interpretation is seen particularly clearly by representing quantum states in terms of probabilities for the outcomes of a fixed, fiducial symmetric informationally complete (SIC) measurement. We further explore the extent to which the general form of state-space structure of quantum mechanics.

1 Introduction: Unperformed Measurements Have No Outcomes

> We choose to examine a phenomenon which is impossible, *absolutely* impossible, to explain in any classical way, and which has in it the heart of quantum mechanics. In reality, it contains the *only* mystery. We cannot make the mystery go away by "explaining" how it works. We will just *tell* you how it works. In telling you how it works we will have told you about the basic peculiarities of all quantum mechanics.
>
> \- R.P Feynman, 1964

These words come from the opening chapter on quantum mechanics in Richard Feynman's famous Feynman Lectures on Physics (Feynman, 1964). With them he plunged into a discussion of the double-slit experiment using individual electrons. Imagine if you will, however, someone well-versed in the quantum foundations debates of the last 30 years - since the Aspect experiment say (Aspect, 1982) - yet naively unaware of when Feynman wrote this.

What might he conclude that Feynman was talking about? Would it be the double-slit experiment? Probably not. To the modern mindset, a good guess would be that Feynman was talking about something to do with quantum entanglement or Bell-inequality violations. In the history of foundational thinking, the double-slit experiment has fallen by the wayside.

So, what is it that quantum entanglement teaches us - via EPR-type considerations and Bell-inequality violations - that the double-slit experiment does not? A common answer is that quantum mechanics does not admit a "local hidden variable" formulation. Too quick and dirty, some would say (Norsen, 2006). However, the conclusion drawn there - that a Bell inequality violation implies the failure of locality, full stop - is based (in part) on taking the EPR criterion of reality or variants of it as sacrosanct. As will become clear in this discussion, we do not take it so. By this one usually means the conjunction of two statements (Bell, 1964, 1981):

(1) that experiments in one region of spacetime cannot instantaneously affect matters of fact at far away regions of spacetime, and

(2) that there exist "hidden variables" that in some way "explain" measured values or their probabilities.

Bell-inequality violations imply that one or the other or some combination of both these statements fails. This, many would say, is the deepest "mystery" of quantum mechanics.

This mystery has two sides. It seems the majority of physicists who care about these things think it is locality (condition 1 above) that has to be abandoned through the force of the experimentally observed Bell-inequality violations - i.e., they think there really are "spooky actions at a distance".

Indeed, it flavors almost everything they think of quantum mechanics, including the *interpretation* of the toy models they use to better understand the theory. Take the recent flurry of work on Popescu-Rohrlich boxes (Popescu, 1994). These are imaginary devices that give rise to greater-than-quantum violations of various Bell inequalities. Importantly, another common name for these devices is the term "nonlocal boxes" (Barrett, 2005). Their exact definition comes via the magnitude of a Bell-inequality violation - which entails the non-pre-existence of values or a violation of locality or both - but the commonly used name opts only to recognize nonlocality. They're not called no-hidden-variable boxes, for instance. The nomenclature is psychologically telling.

Yet, there is a minority that thinks the abandonment of condition 2 is the more warranted conclusion (D'Ariano, 2009; Demopoulos, 2008; Mermin, 1999; Peres, 1978; Plotnitsky, 2006; Wheeler, 1982; Zeilinger, 1996; Zukowski, 2005). Among these are the Quantum Bayesians (Appleby, 2005a,b; Caves, 2002, 2007; Fuchs, 2002, 2003, 2004, 2010,b, 2012; Mermin, 2012; Schack,2001,2003; Timpson,2008).

For alternative developments of several Bayesian-inspired ideas in quantum mechanics, see (Baez, 2003; Bub, 2011; Caticha, 2006; Goyal, 2008; Leifer, 2006, 2007, 2012; Mana, 2007; Pitowsky, 2003, 2005; Rau, 2007; Srednicki, 2005; Warmuth, 2009; Youssef, 2001). We keep these citations separate from the remark above because of various distinctions within each from what we are calling Quantum Bayesianism - these distinctions range from

(1) the particular strains of Bayesianism each adopts, to

(2) whether quantum mechanics is a *generalized* probability theory or rather simply an application within Bayesian probability per se, to

(3) the level of the agent's involvement in bringing about the outcomes of quantum measurements.

There are nonetheless sometimes striking kinships between the ideas of these papers and the effort here, and the papers are well worth studying.

Giving up on hidden variables implies in particular that measured values do not *pre-exist* the act of measurement. A measurement does not merely "read off" the values, but enacts or creates them by the process itself. In a slogan inspired by Asher Peres (Peres, 1978), "unperformed measurements have no outcomes".

Among the various arguments the Quantum Bayesians use to come to this conclusion, not least in importance is a thoroughgoing personalist account of *all* probabilities (Bernardo, 1994; de Finetti, 1931, 1990; Jeffrey, 2004; Ramsey, 1931; Savage, 1954) - where the "all" in this sentence includes probabilities for quantum measurement outcomes and even the probability-1 assignments among these (Caves, 2007). From the Quantum-Bayesian point of view, this is the only sound interpretation of probability. Moreover, this move for quantum probabilities frees up the quantum state from any objectivist obligations. In so doing it wipes out the mystery of quantum-state-change at a distance (Einstein, 1951; Fuchs, 2000; Timpson, 2008) and much of the mystery of wave function collapse as well (Fuchs, 2002, 2010b, 2013).

But what does all this have to do with Feynman? Apparently Feynman too saw something of a truth in the idea that "unperformed measurements have no out-comes". Yet, he did so because of considerations to do with the double-slit experiment. Later in the lecture he wrote,

> Is it true, or is it *not* true that the electron either goes through hole 1 or it goes through hole 2? The only answer that can be given is that we have found from experiment that there is a certain special way that we have to think in order that we do not get into inconsistencies. What we must say (to avoid making wrong predictions) is the following. If one looks at the holes or, more accurately, if one has a piece of apparatus which is capable of determining whether the electrons go through hole 1 or hole 2, then one can say that it goes either through hole 1 or hole 2. *But*, when one does *not* try to tell which way the electron goes, when there is nothing in the experiment to disturb the electrons, then one may *not* say that an electron goes either through hole 1 or hole 2. If one does say that, and starts to make any deductions from the statement, he will make errors in the analysis. This is the logical tightrope on which we must walk if we wish to describe nature successfully.

Returning to the original quote, we are left with the feeling that this is the very thing Feynman saw to be the "basic peculiarity of all quantum mechanics".

One should ask though, is his conclusion really compelled by so simple a phenomenon as the double slit? How could simple "interference" be so far-reaching in its metaphysical implications? Water waves interfere and there is no great mystery there. Most importantly, the double-slit experiment is a story of measurement on a single quantum system, whereas the story of EPR and Bell is that of measurement on *two* seemingly disconnected systems.

Two systems are introduced for a good reason. Without the guarantee of arbitrarily distant parts within the experiment - so that one can conceive of measurements on one, and draw inferences about the other - what justification would one have to think that changing the conditions of the experiment (from one slit closed to both slits open) should *not* make a deep conceptual difference to its analysis? Without such a guarantee for underwriting a belief that some matter of fact stays constant in the consideration of two experiments, one - it might seem - would be quite justified in responding, "Of course, you change an experiment, and you get a different probabil- ity distribution arising from it. So what?".

This is a point Koopman (Koopman, 1957) and Ballentine (Ballentine, 1986) seem happy to stop the discussion with. For instance, Ballentine writes, "One is well advised to beware of probability statements of the form, $P(X)$, instead of $P(X|C)$. The second argument may be safely omitted only if the conditional event or information is clear from the context, and only if it is constant throughout the problem. This is not the case in the double slit experiment. . . . We observe from experiment that $P(X|C_3) \neq P(X|C_1) + P(X|C_2)$. This fact, however, has no bearing on the validity of . . . probability theory".

For quite a long time, the authors thought that Feynman's logical path from example to conclusion - a conclusion that we indeed agree with - was simply unwarranted. The argument just does not seem to hold to the same stringent standards as Bell-inequality analyses.

However, we have recently started to appreciate that there may be something of substance in Feynman's argument nonetheless. It is just not so easily seen without the proper mindset. The key point is that the so-called "interference" in the example is not in a material field - of course it was never purported to be - but in something so ethereal as probability itself (a logical, not a physical, construct). Most particularly, Feynman makes use of a beautiful and novel move: He analyzes the probabilities in an experiment that *will* be done in terms of the probabilities from experiments that *won't* be done. He does not simply conditionalize the probabilities to the two situations and let it go at that. Rather he tries to see the probabilities in the two situations as functions of each other. Not functions of a condition, but functions (or at least relations) of each other. This is an important point. There is no necessity that the world give a relation between these probabilities, yet it does:

Quantum mechanics is what makes the link precise. Feynman seems to have a grasp on the idea that the essence of the quantum mechanical formalism is to provide a tool for analyzing the factual in terms of a counterfactual. In his own case, he then develops the formalism of amplitudes to mediate between the various probabilities, whereas in this discussion we will doggedly stick to probabilities, and only probabilities. It is only his conceptual point that we want to develop, not his technical apparatus.

Here is the way Feynman put it in a paper titled, "The Concept of Probability in Quantum Mechanics", (Feynman, 1951):

> I should say, that in spite of the implication of the title of this talk the concept of probability is not altered in quantum mechanics. When I say the probability of a certain outcome of an experiment is p, I mean the conventional thing, that is, if the experiment is repeated many times one expects that the fraction of those which give the outcome in question is roughly p. I will not be at all concerned with analyzing or defining this concept in more detail, for no departure from the concept used in classical statistics is required.
>
> What is changed, and changed radically, is the method of calculating probabilities.

Far be it from us to completely agree with everything in this quote. For instance, the concept of "probability *as* long-run frequency" is anathema to a Bayesian of any variety (Good, 1983). It is worth pointing out, however, that Feynman was not always consistently a frequentist in his thinking about probability. For instance, in the *Lectures on Physics*, chapter I-6, it says (Feynman, 1964)

> An experimental physicist usually says that an "experimentally determined" probability has an "error", and writes
>
> $$P(H) = \frac{N_H}{N} \pm \frac{1}{2\sqrt{N}}$$
>
> There is an implication in such an expression that there is a "true" or "correct" probability which *could* be computed if we knew enough, and that the observation may be in "error" due

> to a fluctuation. There is, however, no way to make such thinking logically consistent. It is probably better to realize that the probability concept is in a sense subjective, that it is always based on uncertain knowledge, and that its quantitative evaluation is subject to change as we obtain more information.

And B. O. Koopman (Koopman, 1957) is surely right when he says,

> Ever since the advent of modern quantum mechanics in the late 1920's, the idea has been prevalent that the classical laws of probability cease, in some sense, to be valid in the new theory. More or less explicit statements to this effect have been made in large number and by many of the most eminent workers in the new physics. ....

In fact, Koopman is speaking directly of Feynman here. Moreover, both he and Ballentine (Ballentine, 1986) have criticized Feynman on the same point: That with his choice of the word "changed" in the last quote, Feynman implicates himself in not recognizing that the conditions of the three contemplated experiments are distinct and, hence, in not conditionalizing his probabilities appropriately. Thus - Koopman and Ballentine say - it is no wonder Feynman thinks he sees a violation of the laws of probability. In my opinion however, Koopman and Ballentine are hanging too much on the word "changed" - we rather see it as an unfortunate choice of wording on Feynman's part. That he understood that the conditions are different in a deep and inescapable way in the three contemplated experiments, we feel, is documented well enough in the quote above from his 1964 lecture.

> Such a thesis is surprising, to say the least, to anyone holding more or less conventional views regarding the positions of logic, probability, and experimental science: many of us have been apt - perhaps too naively - to assume that experiments can lead to conclusions only when worked up by means of logic and probability, whose laws seem to be on a different level from those of physical science.

But there is a kernel of truth here that should not be dismissed in spite of Feynman's diversion to frequentism and his in-the-end undefensible choice of the word "changed". Paraphrasing Feynman,

> *The concept of probability is not altered in quantum mechanics (it is personalistic Bayesian probability). What is radical is the recipe it gives for calculating new probabilities from old.*

For, quantum mechanics - we plan to show in this discussion - gives a resource that raw Bayesian probability theory does not: It gives a rule for forming probabilities for the outcomes of *factualizable* experiments (experiments that may actually be performed) from the probabilities one assigns for the outcomes of a designated counterfactual experiment (an experiment only imagined, and though possible to do, never actually performed).

We coin this new term because it stands as a better counterpoint to the term "counterfactual" than the term "actualizable" seems to. We also want to capture the following idea a little more plainly: Both measurements being spoken of here are only potential measurements - it is just that one will always be considered in the imaginary realm, whereas the other may one day become a fact of the matter if it is actually performed. So, yes, unperformed measurements have no outcomes as Peres expressed nicely; nonetheless, imagining their performance can aid in analyzing the probabilities one *ought to* assign for an experiment that may factually be per- formed. Putting it more carefully than Feynman: Quantum mechanics does not provide a radical change to the method of calculating probabilities; it provides rather an empirical *addition* to the laws of Bayesian probability.

In this discussion, we offer a modernization and Bayesianification of Feynman's consideration by making intimate use of a potential (we say "potential" because so far the representation has been seen to exist only for finite dimensional quantum systems with dimension ≤ 100. More will be said on this in Section 3) representation of quantum states unknown in his time: It is one based on SICs or symmetric informationally complete observables (Appleby, 2005, 2007; Caves, 1999; Fuchs, 2004b; Renes, 2004; Zauner, 1999). The goal is to make it more transparent than ever that the content of the Born Rule is not that it gives a procedure for *setting* probabilities (from some independent entity called "the quantum state"), but that it represents a "method of calculating probabilities", new ones from old.

That this must be the meaning of the Born Rule more generally for Quantum Bayesianism has been argued from several angles by the authors before (Caves, 2007; Fuchs, 2013). For instance, in (Caves, 2007), we put it this way:

> We have emphasized that one of the arguments often repeated to justify that quantum-mechanical probabilities are objective, rather than subjective, is that they are "determined by physical law". But, what can this mean? Inevitably, what is being invoked is an idea that quantum states $|\psi\rangle$ have an existence independent of the probabilities they give rise to through the Born Rule, $p(i) = \langle\psi| E_i |\psi\rangle$. From the Bayesian perspective, however, these expressions are not independent at all, and what we have argued... is that quantum states are every bit as subjective as any Bayesian probability. What then is the role of the Born Rule? Can it be dispensed with completely?
>
> It seems no, it cannot be dispensed with, even from the Bayesian perspective. But its significance is different than in other developments of quantum foundations: The Born Rule is not a rule for *setting* probabilities, but rather a rule for *transforming* or *relating* them.
>
> For instance, take a complete set of $d+1$ observables $O^k, k = 1, ..., d+1$, for a Hilbert space of dimension d. Subjectively setting probabilities for the d outcomes of each such measurement uniquely determines a quantum state $|\psi\rangle$ (via inverting the Born Rule). Thus, as concerns probabilities for the outcomes of any other quantum measurements, there can be no more freedom. All further probabilities are obtained through linear transformations of the originals. In this way, the role of the Born Rule can be seen as having something of the flavor of Dutch-book coherence, but with an empirical content added on top of bare, law-of-thought probability theory: An agent interacting with the quantum world would be wise to adjust his probabilities for the out- comes of various measurements to those of quantum form if he wants to avoid devastating consequences. The role of physical law - i.e., the assumption that the world is made of quantum mechanical stuff - is codified in how measurement probabilities are related, not how they are set.

What is new in the present discussion is the emphasis on a single designated observable for the counterfactual thinking, as well as a detailed exploration of the rule for combining probabilities in this picture. Particularly, we will see that a significant part of the structure of quantum-state space arises from the consistency of that rule - a single formula we designate the *urgleichung* (German for "primal equation"). The urgleichung is the stand-in and correction in our context for Feynman's not-quite-right but nonetheless suggestive sentence, "What is changed, and changed radically, is the method of calculating probabilities".

Returning to the point we started our discussion with, the one about interference going by the wayside in quantum foundations, we should say the following. To the ex- tent that the full formalism of quantum mechanics can be re-derived from a simple Feynman-style scenario - even if not the double-slit experiment per se, but nonetheless one wherein probabilities for the outcomes of factualizable experiments are obtained from probabilities in a family of designated counterfactual ones - that scenario must indeed express the essence of quantum mechanics. For if these considerations give rise to the full formalism of the theory (Hilbert spaces, positive operators, the possibility of tensor-product decompositions, etc.), they must give rise to entanglement, Bell-inequality violations, and Kochen-Specker 'paradoxes' as well: These will be established as corollaries to the formalism. Hidden in these ostensibly different considerations would be every mystery and every paradox' of quantum mechanics. And if this is truly the case, who could say that the simple scenario does not carry in it the essence of quantum mechanics? From our point of view, it goes without saying that the exploration of quantum mechanics' ability to engender Bell-inequality violations and Kochen-Specker theorems is an immensely instructive activity for sorting out the implications of the theory. Nonetheless, one should not lose sight of the potential loss of understanding that can be incurred by confusing a corollary with a postulate. In this sense, Feynman may well have had the right intuition.

1.1 Outline of the Discussion

The plan of the discussion is as follows. In Section II, we review the personalist Bayesian account of probability, showing how some Dutch-book arguments work, and emphasizing a point we have not seen emphasized before: Bayes' rule and the Law of Total Probability, Eqs. (1) and (3) below, are not necessities in a Bayesian account of probability. These rules are enforceable when there is a *conditional lottery* in the picture that can be gambled upon. But when there is no such lottery, the rules hold no force - without a conditional lottery there is nothing in Dutch-book coherence itself that can be used to compel the rules.

In Section III, we review the notion of a SIC (symmetric informationally complete observable), and show a sense in which it is a very special measurement. Most importantly we delineate the full structure of quantum- state space in SIC terms. It turns out that, by making use of a SIC instead of any other informationally complete measurement, the formalism becomes uniquely simple and compact. We also show that unitary time evolution, when written in SIC terms, looks (formally at least) almost identical to classical stochastic evolution.

Section IV contains the heart of the discussion. In it, we introduce the idea of thinking of an imaginary (counterfactual) SIC behind all quantum measurements, so as to give an imaginary conditional lottery with which to define conditional probabilities. We then show how to write the Born Rule in these terms, and find it to appear strikingly similar to the Law of Total Probability, Eq. (3). We then note how this move in interpretation is radically different from the one offered by the followers of "objective chance" in the Lewisian sense (Lewis, 1986a,b).

In Section V, we show that one can derive some of the features of quantum-state space by taking this modified or *Quantum* Law of Total Probability as a postulate. Particularly, we show that with a small number of further assumptions, it gives rise to a generalized Bloch sphere and seems to define an underlying "dimensionality" for a system that matches the one given by its quantum mechanical state space. We also demonstrate other features of the geometry these considerations give rise to.

In Section VI, we step back further in our considerations and explore the extent to which the particular constants d^2, $d+1$, and $1/d$ in our Quantum Law of Total Probability, Eq. (64), can arise from more elementary considerations. This section is a preliminary attempt to understand the origin of the equation treated simply as a postulate in Section V.

In Section VII we give a brief discussion of where we stand at this stage of research. Finally in Section VIII, we close the paper by discussing how our work is still far from done: Hilbert space, from a Quantum-Bayesian view, has not yet been derived, only indicated. Nonetheless the progress made here gives us hope that we are inching our way toward a formal expression of the ontology underlying a Quantum-Bayesian vision of quantum mechanics: It has to do with the Peres slogan "unperformed measurements have no outcomes", but tempered with a kind of 'realism' that he probably would not have accepted forthrightly. We say this because of Peres's openly acknowledged positivist tendencies. Peres would sometimes call himself a "recalcitrant positivist". Also see the opening remarks of (Peres, 2005) for a good laugh. On the other hand, it is not a 'realism' that we expect to be immediately accepted by most modern philosophers of science either. See (Dennett, 2004; Nagel, 1989; Price, 1997) for introductions to the "view from nowhere" and "view from nowhen" *weltanschaungen*. This is because it is already clear that whatever it will ultimately turn out to be, it is based on

(a) a rejection of the ontology of the block universe (James, 1882, 1884, 1910), and

(b) a rejection of the ontology of the detached observer (Gieser, 2005; Laurikainen, 1988; Pauli, 1994).

The 'realism' in vogue in philosophy-of-science circles, which makes heavy use of both these elements, is, as Wolfgang Pauli once said, "too narrow a concept" for our purposes (Pauli, 1994). Reality, the stuff of which the world is made, the stuff that was here before agents and observers, is more interesting than that.

2 Personalist Bayesian Probability

From the Bayesian point of view, probability is degree of belief as measured by action. More precisely, we say one has (explicitly or implicitly) assigned a probability $p(A)$ to an event A if, before knowing the value of A, one is willing to either *buy* or *sell* a lottery ticket of the form

$$\boxed{\text{Worth \$1 if } A}$$

for an amount $\$p(A)$. In other words, the personalist Bayesian agent regards $\$p(A)$ as the fair price of the lottery ticket. He would regard it as advantageous to buy it for any price less than $\$p(A)$, or to sell it for any price greater than $\$p(A)$. The personalist Bayesian position adds only that this is the full meaning of probability; it is nothing more and nothing less than this definition. Particularly, nothing intrinsic to the event or proposition A can help declare $p(A)$ right or wrong, or more or less rational. The value $p(A)$ is solely a statement about the agent who assigns it.

Nonetheless, even for a personalist Bayesian, probabilities do not wave in the wind. Probabilities are held together by a normative principle: That whenever an agent declares probabilities for various events - say $A \neg B$ (" not B"), $A \vee B$ ("A or B"), $A \wedge C$ ("A and C"), etc. - he should *strive* to never gamble (i.e., buy and sell lottery tickets) so as to incur what he believes will be a sure loss. This normative principle is known as Dutch- book coherence. And from it, one can derive the usual calculus of probability theory.

This package of views about probability (that in value it is personal, but that in function it is akin to the laws of logic) had its origin in the mid-1920s and early 1930s with the work of F. P. Ramsey (Ramsey, 1931) and B. de Finetti (de Finetti, 1931). J. M. Keynes characterizes Ramsey's position succinctly (Keynes, 1951):

> [Ramsey] succeeds in showing that the calculus of probabilities simply amounts to a set of rules for ensuring that the system of degrees of belief which we hold shall be a consistent system. Thus the calculus of probabilities belongs to formal logic. But the basis of our degrees of belief - or the a priori, as they used to be called - is part of our human outfit, perhaps given us merely by natural selection, analogous to our perceptions and our memories rather than to formal logic.

And B. O. Koopman writes (Koopman, 1940):

> The intuitive thesis in probability holds that both in its meanings and in the laws which it obeys, probability derives directly from the intuition, and is prior to objective experience; it holds that it is for experience to be interpreted in terms of probability and not for probability to be interpreted in terms of experience . . .

Let us go through some of the derivation of the probability calculus from Dutch-book coherence so that we may better make a point concerning quantum mechanics afterward. Here we basically follow the development in Richard Jeffrey's posthumously published book Subjective Probability, The Real Thing (Jeffrey, 2004; Skyrms, 1987), but with our own emphasis.

First we establish that our normative principle requires $0 \leq P(A) \leq 1$. For supposes $P(A) < 0$. This means an agent will sell a ticket for negative money - i.e., he will pay someone $\$p(A)$ to take the ticket off his hands. Regardless of whether A occurs or not, the agent will then be sure he will lose money. This violates the normative principle. Now, take the case $P(A) > 1$. This means the agent will buy a ticket for more than it is worth even in the best case - again a sure loss for him and a violation of the normative principle. So, probability in the sense of ticket pricing should obey the usual range of values.

Now let us establish the probability sum rule. Suppose our agent believes two events A and B to be mutually exclusive - i.e., he is sure that if A occurs, B will not, or if B occurs, A will not. We can contemplate three distinct lottery tickets:

$$\boxed{\text{Worth \$1 if } A \vee B}$$

and

$$\boxed{\text{Worth \$1 if } A} \quad \boxed{\text{Worth \$1 if } B}$$

Clearly the value of the first ticket should be the same as the total value of the other two. For instance, suppose an agent had set $P(A \vee B)$, $P(A)$ and $P(B)$ such that $P(A \vee B) > P(A) + P(B)$. Then -by definition - when confronted with a seller of the first ticket, he must be willing to buy it, and when confronted with a buyer of the other two tickets, he must be willing to sell them. But then the agent's initial balance sheet would be negative: $-\$P(A \vee B) + \$P(A) + \$P(B) < \$0$. And whether A or B or neither event occurs, it would not improve his finances: If a dollar flows in (because of the bought ticket), it will also flow out (because of the agent's responsibilities for the sold tickets), and still the balance sheet is negative. The agent is sure of a loss. A similar argument goes through if the agent had set his ticket prices so that $P(A \vee B) < P(A) + P(B)$. Thus whatever values are set, the normative principle prescribes that it had better be the case that $P(A \vee B) = P(A) + P(B)$.

Consider now the following lottery ticket of a slightly different structure:

$$\boxed{\text{Worth } \$\frac{m}{n} \text{ if } A}$$

where $m \leq n$ are integers. Does Dutch-book coherence say anything about the value of this ticket in comparison to the value of the standard ticket - i.e., one worth \$1 if A? It does. An argument quite like the one above dictates that it should be valued $\$t\frac{m}{n}$ if A. If a real number α were in place of $\frac{m}{n}$ a similar result follows from a limiting argument.

Now we come to the most interesting and important case. Bayesian probability is not called by its name for lack of a good reason. A key rule in probability theory is Bayes' rule relating joint to conditional probabilities:

$$p(A \wedge B) = p(A)p(B|A) \tag{1}$$

Like the rest of the structure of probability theory within the Bayesian conception, this rule must arise from an application of Dutch-book coherence. What is that application?

The only way anyone has seen how to do it is to introduce the idea of a *conditional lottery* (Kyburg, 1980). In such a lottery, the value of the event A is revealed to the agent first. If A obtains, the lottery proceeds to the revelation of event B, and finally all monies are settled up. If on the other hand $\neg A$ obtains, the remainder of the lottery is called off, and the monies put down for any "conditional tickets" are returned. That is to say, the meaning of $p(B|A)$ is taken to be the price $\$p(B|A)$ at which one is willing to *buy* or *sell* a lottery ticket of the following form:

$$\boxed{\text{Worth \; if } A \wedge B. \quad \text{But return price if } \neg A}$$

for price $\$p(B|A)$. Explicitly inserting the definition of $p(B|A)$, this becomes:

$$\boxed{\text{Worth \; if } A \wedge B. \quad \text{But return } \$p(B|A) \text{ if } \neg A}$$

Now comes the coherence argument. For, if you think about it, the price for this ticket had better be the same as the total price for these two tickets:

$$\boxed{\text{Worth \$1 if } A \wedge B}$$

$$\boxed{\text{Worth } \$p(B|A) \text{ if } \neg A}$$

That is to say, to guard against a sure loss, we must have

$$\begin{aligned} p(B|A) &= p(A \wedge B) + p(B|A)p(\neg A) \\ &= p(A \wedge B) + p(B|A) - p(B|A)p(A) \end{aligned} \tag{2}$$

Consequently, Eq. (1) should hold whenever there is a conditional lottery under consideration.

2.1 When a Conditional Lottery is Not Without Consequence

But what if the conditional lottery is called off because the draw that was to give rise to the event A does not take place? In this case the probabilities $p(A)$ and $p(B|A)$ refer to a counterfactual, and there is no reason to assume the validity of Eq. (1).

It is worth investigating the idea of counterfactuals in some more detail. Suppose an agent makes a measurement of a variable X that takes on values x, followed by a measurement of a variable Y with mutually exclusive values y. A Dutch bookie asks him to commit on various unconditional and conditional lottery tickets. What can we say of the probabilities he ought to ascribe? A minor variation of the Dutch-book arguments above tells us that whatever values of $p(x)$, $p(y)$, and $p(y|x)$ he commits to, they ought - if he is coherent - satisfy the Law of Total Probability:

$$p(y) = \sum_x p(x)p(y|x) \tag{3}$$

Imagine now that the X measurement is called off, so there will only be the Y measurement. Is the agent still normatively committed to buying and selling Y-lottery tickets for the price $\$p(y)$ in Eq. (3) that he originally expressed? Not at all! That would clearly be silly in some situations, and no clear-headed Bayesian would ever imagine otherwise. The action bringing about the result of the X measurement might so change the situation for bringing about Y that he simply would not gamble on it in the same way. To hold fast to the $\$p(y)$ valuation of a Y-lottery ticket, then, is not a necessity enforced by coherence, but a judgment that might or might not be the right thing to do.

In fact, one might regard holding fast to the initial value $\$p(y)$ in spite of the nullification of the conditional lottery as the formal definition of precisely what it means to judge an unperformed measurement to have an outcome. It means one judges that looking at the value of X is incidental to the whole affair, and this is reflected in the way one gambles on Y (Fuchs, 2012). So, if $q(y)$ represents the probabilities with which the agent gambles supposing the X-lottery nullified, then a formal statement of the Peresian slogan that the unperformed X measurement had no outcome (i.e., measuring X matters, and it matters even if one makes no note of the outcome) is that

$$q(y) \neq p(y) \tag{4}$$

Still, one might imagine situations in which even if an agent judges that equality does not hold for them, he nonetheless judges that $q(y)$ and $p(y)$ should bear a precise relation to each other. In Section 4, we will show that, in fact, the positive content of the Born Rule as an addition to Bayesianism is to connect the probabilities for two measurements, one factual and one counterfactual, for which Dutch-book coherence alone does not provide a precise relationship.

3 Expressing Quantum-State Space in Terms of SICs

Let $\mathcal{H}_d$ be a finite d-dimensional Hilbert space associated with some physical system. A quantum state for the system is usually expressed as a unit-trace positive semi-definite linear operator $\rho \in \mathcal{L}(\mathcal{H}_d)$. However, through the use of a *symmetric informationally complete observable* (or SIC) as a reference observable, we can find a rather elegant representation of quantum states directly in terms of an associated set of probability distributions.

A SIC is an example of a generalized measurement or positive operator-valued measure (POVM) (Peres, 1993). A POVM is a collection $\{E_i\}, i = 1, ..., n$, of positive semi-definite operators E_i on $\mathcal{H}_d$ such that

$$\sum_i E_i = I \tag{5}$$

where n is in general unrelated to d and may be larger or smaller than d. Supposing a quantum state ρ, the probability of the measurement outcome labeled i is then given by

$$p(i) = tr\, \rho E_i \tag{6}$$

The POVMs represent the most general kinds of quantum measurement that can be made on a system. A von Neumann measurement is a special POVM where the E_i are mutually orthogonal projection operators. Mathematically, any POVM can be written as a unitary interaction with an ancillary quantum system, followed by a von Neumann measurement on the ancillary system (Nielsen, 2000).

We will use a pseudo-Dirac notation $\| v \rangle\!\rangle$ for vectors in a *real* vector space of d^2 dimensions. The relevant probability simplex for us - the one we are mapping quantum states ρ to, denoted Δ_{d^2} - is a convex body within this linear vector space. Thus, its points may be expressed with the notation $\| p \rangle\!\rangle$ as well. The choice of a pseudo-Dirac notation for probability distributions also emphasizes that one should think of the valid $\| p \rangle\!\rangle$ as a direct expression of the set of quantum states.

We can provide an injective mapping between the convex set of density operators and the set of probability distributions

$$\| p \rangle\!\rangle = \big(p(1), p(2), ..., p(d^2)\big)^T \tag{7}$$

over d^2 outcomes - the probability simplex Δ_{d^2} - by first fixing any so-called minimal informationally complete fiducial measurement $\{E_i\}, i = 1, ..., d^2$. This is a POVM with all the E_i linearly independent. With respect to such a measurement, the probabilities $p(i)$ for its outcomes completely specify ρ. This follows because the E_i form a basis for $\mathcal{L}(\mathcal{H}_d)$, and the probabilities $p(i) = tr\, \rho E_i$ can be viewed as instances of the Hilbert-Schmidt inner product

$$(A, B) = tr\, A^\dagger B \tag{8}$$

The quantities $p(i)$ thus merely express the projections of the vector ρ onto the basis vectors E_i. These projections completely fix the vector ρ.

One can see how to calculate ρ in terms of the vector $\| p \rangle\!\rangle$ in the following way. Since the E_i form a basis, there must be some expansion

$$\rho = \sum_j \alpha_j E_j \tag{9}$$

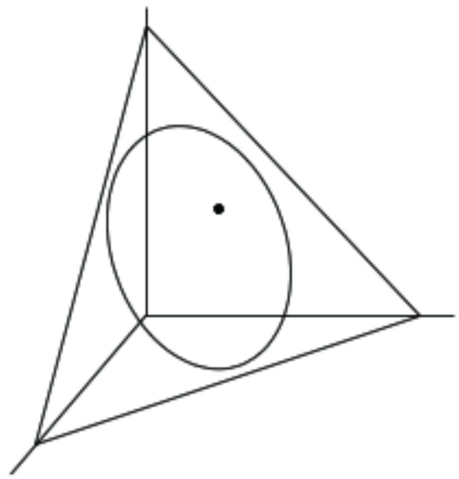

Figure 1: The planar surface represents the convex set of all probability distributions over d^2 outcomes - the probability simplex Δ_{d^2}. The probability distributions valid for representing the set of quantum states, however, is a smaller convex set within the simplex - here depicted as an ellipsoid. In actual fact, however, the convex shape is quite complex. The choice of a SIC for defining the mapping makes the shape as simple as it can be with respect to the natural geometry of the simplex.

where the α_j are real numbers making up a vector $\|\alpha\rangle\!\rangle$. Thus,

$$p(i) = \sum_j \alpha_j tr\, E_i E_J \tag{10}$$

If we let a matrix M be defined by entries

$$M_{ij} = tr\, E_i E_j \tag{11}$$

this just becomes

$$\|p\rangle\!\rangle = M\|\alpha\rangle\!\rangle \tag{12}$$

Using the fact that M is invertible because the E_i are linearly independent, we have finally

$$\|\alpha\rangle\!\rangle = M^{-1}\|p\rangle\!\rangle \tag{13}$$

The most important point of this exercise is that with such a mapping established, one has every right to think of a quantum state as a probability distribution *full stop*. In (Fuchs, 2002) it is argued that, conceptually, it is in- deed nothing more. However, it is important to note that the mapping $\rho \mapsto \| p \rangle\rangle$, though injective, cannot be surjective. In other words, only some probability distributions in the simplex are valid for representing quantum states. A significant part of understanding quantum mechanics is understanding the range of shapes available under these mappings (Bengtsson, 2006).

Particularly, it is important to recognize that informationally complete measurements aboundthey come in all forms and sizes. Hence there is no unique representation of this variety for quantum states. A reasonable question thus follows: What is the best measurement one can use for a mapping $\rho \mapsto \| p \rangle\rangle$? One would not want to unduly burden the representation with extra terms and calculations if one does not have to. For instance, it would be beautiful if one could take the informationally complete measurement $\{E_i\}$ so that M is simply a diagonal matrix or even the identity matrix itself. Such an extreme simplification, however, is not in the cards - it cannot be done.

Its failure does, however, point to an interesting direction worthy of development: If one cannot make M diagonal, one might still want to make M as close to the identity as possible. A convenient measure for how far M is from the identity is the squared Frobenius distance:

$$\begin{aligned} F &= \sum_{ij} (\delta_{ij} - M_{ij})^2 \\ &= \sum_i \left(1 - tr\, E_i^2\right)^2 + \sum_{i \neq j} (tr\, E_i E_j)^2 \end{aligned} \tag{14}$$

We can place a lower bound on this quantity with the help of a special instance of the Schwarz inequality: If λ_r is any set of n nonnegative numbers, then

$$\sum_r \lambda_r^2 \geq \frac{1}{n} \left(\sum_r \lambda_r \right)^2 \tag{15}$$

with equality if and only if $\lambda_1 = ... = \lambda_n$. Thus,

$$\begin{aligned} F \geq & \frac{1}{d^2} \left(\sum_i (1 - tr\, E_i^2) \right)^2 \\ &+ \frac{1}{d^4 - d^2} \left(\sum_{i \neq j} tr\, E_i E_j \right)^2 \end{aligned} \tag{16}$$

Equality holds in this if and only if there are constants m and n such that $tr\, E_i^2 = m$ for all i and $E_iE_j = n$ for all $i \neq j$. Since

$$d = tr\, I^2 = \sum_{ij} tr\, E_iE_j = \sum_i tr\, E_i^2 + \sum_{i \neq j} tr\, E_iE_j \tag{17}$$

m and n must be related by

$$m + (d^2 - 1)n = \frac{1}{d} \tag{18}$$

On the other hand, with these conditions fulfilled the E_i must all have the same trace. For,

$$tr\, E_k = \sum_i tr\, E_kE_i = m + (d^2 - 1)n \tag{19}$$

Consequently

$$tr\, E_k = \frac{1}{d} \tag{20}$$

Now, how large can m be? Take a positive semi-definite matrix A with $tr\, A = 1$ and eigenvalues λ_i. The $\lambda_i \leq 1$, and clearly $tr\, A^2 \leq tr\, A$ with equality if and only if the largest λ_i is equal to 1. Hence, dE_k will give the largest allowed value m if

$$E_i = \frac{1}{d}\Pi_1 \text{ where } \Pi_i = |\psi_i\rangle\langle\psi_i| \tag{21}$$

for some rank-1 projection operator Π_i. If this obtains, n takes the form

$$n = \frac{1}{d^2(d+1)} \tag{22}$$

In total we have shown that a measurement $\{E_i\}, i = 1, ..., d^2$, 1ill achieve the best lower bound for F if and only if

$$E_i = \frac{1}{d}\Pi_1 \tag{23}$$

with

$$tr\, \Pi_i\Pi_j = \frac{d\delta_{ij} + 1}{d + 1} \tag{24}$$

Significantly, it turns out that measurements of this variety also have the property of being necessarily informationally complete (Caves, 1999). Let us show this for completeness: It is just a matter of proving that the E_i are linearly independent. Suppose there are some numbers α_i such that

$$\sum_i \alpha_i E_i = 0 \tag{25}$$

Taking the trace of this equation, we infer that $\sum_i \alpha_i = 0$. Now multiply Eq. (25) by an arbitrary E_k and take thew trace of the result. We get

$$\frac{1}{d^2}\sum_i \alpha_i \frac{d\delta_{ij}+1}{d+1} = 0 \tag{26}$$

In other words

$$\sum_i \alpha_i \delta_{ik} = 0 \tag{27}$$

which of course implies $\alpha_k = 0$. So the E_i are linearly independent.

These kinds of measurements are presently a hot topic of study in quantum information theory, and have come to be known as "symmetric informationally complete" quantum measurements (Caves, 1999). As such, the measurement $\{E_i\}$, the associated set of projection operators $\{\Pi_i\}$, and even the set $\{|\psi_i\rangle\}$ are often called SIC (pronounced "seek"). This choice of pronunciation is meant to be in accord with the "pedant's pronunciation" of the Latin adverb *sic* (Bennett, 2008). Moreover, it alleviates any potential confusion between the pluralized form SICs and the number six in conversation. We will adopt that terminology here.

Here is an example of a SIC in dimension-2, expressed in terms of the Pauli operators:

$$\begin{aligned}
\Pi_1 &= \frac{1}{2}\left(I + \frac{1}{\sqrt{3}}(\sigma_x + \sigma_y + \sigma_z)\right)\\
\Pi_2 &= \frac{1}{2}\left(I + \frac{1}{\sqrt{3}}(\sigma_x - \sigma_y - \sigma_z)\right)\\
\Pi_3 &= \frac{1}{2}\left(I + \frac{1}{\sqrt{3}}(-\sigma_x - \sigma_y + \sigma_z)\right)\\
\Pi_4 &= \frac{1}{2}\left(I + \frac{1}{\sqrt{3}}(-\sigma_x + \sigma_y - \sigma_z)\right)
\end{aligned} \tag{28}$$

And here is an example of a SIC in dimension-3 (Tabia, 2012). Taking $\omega = e^{2\pi i/3}$ to be a third-root of unity and $\bar{\omega}$ to be its complex conjugate, let

$$
\begin{aligned}
|\psi_1\rangle &= \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \ |\psi_2\rangle = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \ |\psi_3\rangle = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \\
|\psi_4\rangle &= \begin{pmatrix} 0 \\ \omega \\ -\bar{\omega} \end{pmatrix}, \ |\psi_5\rangle = \begin{pmatrix} -1 \\ 0 \\ \bar{\omega} \end{pmatrix}, \ |\psi_6\rangle = \begin{pmatrix} 1 \\ -\omega \\ 0 \end{pmatrix} \\
|\psi_7\rangle &= \begin{pmatrix} 0 \\ \bar{\omega} \\ -\omega \end{pmatrix}, \ |\psi_8\rangle = \begin{pmatrix} -1 \\ 0 \\ \omega \end{pmatrix}, \ |\psi_9\rangle = \begin{pmatrix} 1 \\ -\bar{\omega} \\ 0 \end{pmatrix}
\end{aligned}
\tag{29}
$$

be defined up to normalization. One can check by quick inspection that (after normalization) these vectors do indeed satisfy Eq. (24).

Do SICs exist for every finite dimension d? Despite many efforts in the last 14 years - see (Appleby, 2005, 2007; Caves, 1999; Fuchs, 2004b; Renes, 2004; Zauner, 1999) and particularly the extensive reference lists in (Scott, 2010) and (Appleby, 2012) - no one presently knows. However, there is a strong feeling in the community that they do, as analytical proofs have been obtained for all dimensions $d = 2-16, 19, 24, 28, 31, 35, 37, 43,$ and 48[Dimensions 2-5 were published in (Zauner, 1999). Dimensions 6-16, 24, 28, 35, and 48 are due to M. Grassl in various publications; see (Scott, 2010). Dimensions 7, 19, 31, 37, and 43 are due to D. M. Appleby, with the latter three as yet unpublished (Appleby, 2012)], and within a numerical precision of 10^{-38}, they have been observed by computational means (Scott, 2010) in all dimensions $d = 2-67$. To lesser numerical precision E. Schnetter has also found SICs in $d = 68-73, 75-81, 83, 84, 89, 93, 100$ (Schnetter, 2012).

The SIC structure is ideal for revealing the essence of the Born Rule as an addition to Dutch-book coherence. From here out, we will proceed as if SICs do indeed always exist. The foundational significance of the following technical development rests on this mathematical conjecture.

Let us spell out in some detail what the set of quantum states written as SIC probability vectors $\|p\rangle\rangle$ looks like. Perhaps the most remarkable thing about a SIC is the level of simplicity it lends to Eq. (13). On top of the theoretical justification that SICs are as near as possible to an orthonormal basis, from

Eqs. (6), (23) and (24) one gets the simple expression (Caves, 2002b; Fuchs, 2004b)

$$\rho = \sum_i \left((d+1)p(i) - \frac{1}{d} \right) \Pi_i \tag{30}$$

In other words, effectively all explicit reference to the matrix M^{-1} disappears from the expression. The components $(d+1)p(i) - \frac{1}{d}$ are obtained by a universal scalar readjustment from the probabilities $p(i)$. This will have important implications.

Still, one cannot place just any probability distribution $p(i)$ into Eq. (30) and expect to obtain a positive semi-definite ρ. Only some probability vectors $\| p \rangle\!\rangle$ are *valid* ones. Which ones are they? For instance, $p(i) \leq \frac{1}{d}$ must be the case, as dictated by Eqs. (6) and (20), and this already restricts the class of valid probability assignments. But there are more requirements than that.

In preparation for characterizing the set of valid probability vectors $\| p \rangle\!\rangle$, let us note that since the Π_k form a basis on the space of operators, we can define operator multiplication in terms of them. This is done by introducing the so-called structure coefficients α_{ijk} for the algebra

$$\Pi_i \Pi_j = \sum_k \alpha_{ijk} \Pi_k \tag{31}$$

A couple of properties follow immediately. Taking the trace of both sides of Eq. (31), one has

$$\sum_k \alpha_{ijk} = \frac{d\delta_{ij} + 1}{d+1} \tag{32}$$

Using this, one gets straightforwardly that

$$tr(\Pi_i \Pi_j \Pi_k) = \frac{1}{d+1} \left(d\alpha_{ijk} + \frac{d\delta_{ij} + 1}{d+1} \right) \tag{33}$$

In other words,

$$\alpha_{ijk} = \frac{1}{d} \left((d+1) tr(\Pi_i \Pi_j \Pi_k) - \frac{d\delta_{ij} + 1}{d+1} \right) \tag{34}$$

For the analogue of Eq. (32) but with summation over the first or second index, one gets,

$$\sum_i \alpha_{ijk} = d\delta_{jk} \quad \text{and} \quad \sum_j \alpha_{ijk} = d\delta_{ik} \tag{35}$$

With these expressions in hand, one sees a very direct connection between the structure of the algebra of quantum states when written in operator language and the structure of quantum states when written in probability-vector language. For, the complete convex set of quantum states is fixed by the set of its extreme points, i.e., the pure quantum states - rank-1 projection operators. To characterize this set algebraically, one method is to note that these are the only hermitian operators satisfying $\rho^2 = \rho$. Using Eq. (30), we find that a quantum state $\| p\rangle\!\rangle$ is pure if and only if its components satisfy these d^2 simultaneous quadratic equations:

$$p(k) = \frac{1}{3}(d+1)\sum_{ij}\alpha_{ijk}p(i)p(j) + \frac{2}{3d(d+1)} \tag{36}$$

Another way to characterize this algebraic variety - an algebraic variety is defined as the set of solutions of a system of polynomial equations - is to make use of a theorem of Flammia, Jones, and Linden (Flammia, 2004; Jones, 2005): A hermitian operator A is a rank-one projection operator if and only if $tr\,A^2 = tr\,A^3 = 1$. The theorem is nearly trivial to prove once one's attention is drawn to it: Since A it has a real eigenvalue spectrum λ_i. From the first condition, one has that $\sum_i \lambda_i^2 = 1$; from the second, $\sum_i \lambda_i^3 = 1$. The first condition, however, implies that $|\lambda_i| \le 1$ for all i. Consequently $1 - \lambda_i \ge 0$ for all i. Now taking the difference of the two conditions, one sees that $\sum_i \lambda_i^2(1-\lambda_i) = 0$. In order for this to obtain, it must be the case that λ_i is always 0 or 1 exclusively. That there is only one nonzero eigenvalue then follows from using the first condition again. Thus the theorem is proved. However, it seems not to have been widely recognized previous to 2004-2005. So in fact our d^2 simultaneous quadratic equations reduce to just two equations instead, one a quadratic and one a cubic:

$$\sum_i p(i)^2 = \frac{2}{d(d+1)} \tag{37}$$

and

$$\sum_i \alpha_{ijk}p(i)p(j)p(k) = \frac{4}{d(d+1)^2} \tag{38}$$

Note that Eqs. (36) and (38) are complex equations, but one could symmetrize them and make them purely real if one wanted to.

There are also some advantages to working out these equations more explicitly in terms of the completely symmetric 3-index tensor

$$c_{ijk} = \mathrm{Re}\, tr(\Pi_i \Pi_j \Pi_k) \tag{39}$$

In terms of these quantities, the analogues of Eqs. (36) and (38) become

$$p(k) = \frac{(d+1)^2}{3d} \sum_{ij} c_{ijk} p(i) p(j) - \frac{1}{3d} \tag{40}$$

and

$$\sum_i c_{ijk} p(i) p(j) p(k) = \frac{d+7}{(d+1)^3} \tag{41}$$

respectively. The reason for noting this comes from the simplicity of the d^2 matrices C_k with matrix entries $(C_k)_{ij} = c_{ijk}$ from Eq. (39), which is explored in great detail in (Appleby, 2011). To give a flavor of the results, we note for instance that, for each value of k, C_k turns out to have the form (Appleby, 2011)

$$C_k = \| m_k \rangle\!\rangle \langle\!\langle m_k \| + \frac{d}{2(d+1)} Q_k \tag{42}$$

where the k-th vector $\| m_k \rangle\!\rangle$ is defined by

$$\| m_k \rangle\!\rangle = \left(\frac{1}{d+1}, ..., 1, ..., \frac{1}{d+1} \right)^T \tag{43}$$

and Q_k is a $(2d-2)$-dimensional projection operator on the real vector space embedding the probability simplex Δ_{d^2}. Furthermore, using this, one obtains a useful expression for the pure states; they are probabilities satisfying a simple class of quadratic equations

$$p(k) = dp(k)^2 + \frac{1}{2}(d+1) \langle\!\langle p \| Q_k \| p \rangle\!\rangle \tag{44}$$

With Eqs. (37), (40), and (41) we have now discussed the extreme points of the convex set of quantum states - the pure states. The remainder of the set of quantum states is then constructed by taking convex combinations of the pure states. This is an implicit expression of quantum-state space. But SICs can also help give an explicit parameterization of the convex set.

We can see this by starting not with density operators, but with “square roots” of density operators. This is useful because a matrix ρ is positive semi-definite if and only if it can be written as $\rho = B^2$ for some hermitian B. Thus, let

$$B = \sum_i b_i \Pi_i \tag{45}$$

with b_i a set of real numbers. Then,

$$\rho = \sum_k \left(\sum_{ij} b_i b_j \alpha_{ijk} \right) \Pi_k \tag{46}$$

will represent a density operator so long as $tr\, \rho = 1$. This condition requires simply that

$$\left(\sum_i b_i \right)^2 + d \sum_i b_i^2 = d + 1 \tag{47}$$

so that the vectors $(b_1, ..., b_{d^2})$ lie on the surface of an ellipsoid.

Putting these ingredients together with Eq. (6), we have the following parameterization of valid probability vectors $\|p\rangle\!\rangle$:

$$p(k) = \frac{1}{d} \sum_{ij} c_{ijk} b_i b_j \tag{48}$$

Here the c_{ink} are the triple-product constants defined in Eq. (39) and the b_i satisfy the constraint (47).

Finally, let us note what the Hilbert-Schmidt inner product of two quantum states looks like in SIC terms. If a quantum state ρ is mapped to $\|p\rangle\!\rangle$ via a SIC, and a quantum state σ is mapped to $\|q\rangle\!\rangle$, then

$$\begin{aligned} tr\, \rho\sigma &= d(d+1) \sum_i p(i) q(i) - 1 \\ &= d(d+1) \langle\!\langle p \| q \rangle\!\rangle - 1 \end{aligned} \tag{49}$$

Notice a particular consequence of this: Since $tr\, \rho\sigma \geq 0$, the distributions associated with distinct quantum states can never be too nonoverlapping:

$$\langle\!\langle p \| q \rangle\!\rangle \geq \frac{1}{d(d+1)} \tag{50}$$

With this development we have given a broad outline of the shape of quantum-state space in SIC terms. We do this because that shape is our target. Particularly, we are obliged to answer the following question: If one takes the view that quantum states are *nothing more* than probability distributions with the restrictions (48) and (47), what could motivate that restriction? That is, what could motivate it *other than* knowing the usual for- malism for quantum mechanics? The answer has to do with rewriting the Born Rule in terms of SICs, which we will do in Section 4.

3.1 Aside on Unitarity

Let us take a moment to move beyond statics and rewrite quantum dynamics in SIC terms: We do this be- cause the result will have a striking resemblance to the Born Rule itself, once developed in the next section.

Suppose we start with a density operator ρ and let it evolve under unitary time evolution to a new density operator $\sigma = U\rho U^{\dagger}$. If ρ has a representation $p(i)$ with respect to a certain given SIC, σ will have a SIC representation as well - ket us call it $q(j)$. We use the different index j (contrasting with i) to help indicate that we are talking about the quantum system at a later time than the original.

What is the form of the mapping that takes $\|p\rangle\!\rangle$ to $\|q\rangle\!\rangle$? It is simple enough to find with the help of Eqs. (23) and (30):

$$
\begin{aligned}
q(j) &= \frac{1}{d} tr\, \sigma \Pi_j \\
&= \frac{1}{d} \sum_i \left((d+1)p(i) - \frac{1}{d} \right) tr(U\Pi_i U^{\dagger}\Pi_j)
\end{aligned}
\tag{51}
$$

If we now define

$$
r_U(j|i) = \frac{1}{d} tr(U\Pi_i U^{\dagger}\Pi_j) \tag{52}
$$

and remember, e.g., Eq. (21), we have that

$$
0 \le r_U(j|i) \le 1 \tag{53}
$$

and

$$
\sum_j r_U(j|i) = 1 \quad \forall i \quad \text{and} \quad \sum_i r_U(j|i) = 1 \quad \forall j \tag{54}
$$

In other words, the $d^2 \times d^2$ matrix $[r_U(j|i)]$ is a doubly stochastic matrix (Horn, 1985).

Most importantly, one has

$$q(j) = (d+1)\sum_{i=1}^{d^2} p(i) r_U(j|i) - \frac{1}{d} \tag{55}$$

Without the $(d+1)$ factor and the $\frac{1}{d}$ term, this equation would represent classical stochastic evolution. Unitary time evolution in a SIC representation is thus formally close to classical stochastic evolution. As we shall shortly see, this teaches us something about unitarity and its connection to the Born Rule itself.

4 Expressing the Born Rule in Terms of SICs

In this section we come to the heart of the paper: We rewrite the Born Rule in terms of SICs. It is easy enough; we just use the expansion in Eq. (30). Let us first do it for an *arbitrary* von Neumann measurement - that is, any measurement specified by a set of rank-1 projection operators $P_j = |j\rangle\langle j|, j = 1, ..., d$. Expressing the Born Rule the usual way, we obtain these probabilities for the measurement outcomes:

$$q(j) = tr\, \rho P_j \tag{56}$$

Then, by defining

$$r(j|i) = tr\, \Pi_i P_j \tag{57}$$

one sees that the Born Rule becomes

$$q(j) = (d+1)\sum_{i=1}^{d^2} p(i) r(j|i) - 1 \tag{58}$$

Let us take a moment to seek out a good interpretation of this equation. It should be viewed as a direct expression of the considerations laid out in Section 2. For imagine that before performing the P_j measurement - we will call it the "measurement on the ground" - we were to perform a SIC measurement Π_i. We will call the latter the "measurement in the sky".

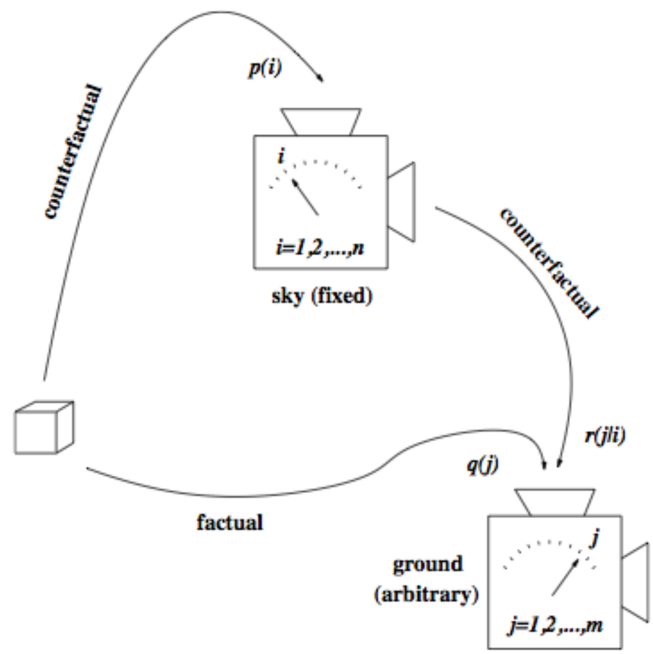

Figure 2: The diagram above expresses the basic conceptual apparatus of this paper. The measurement on the ground, with outcomes $j = 1, ..., m$, is some potential measurement that could be performed in the laboratory - i.e., one that could be factualized. The measurement in the sky, on the other hand, with outcomes $i = 1, ..., n$, is a fixed measurement one can contemplate independently. The probability distributions $p(i)$ and $r(j|i)$ represent how an agent would gamble if a conditional lottery based on the measurement in the sky were operative. The probability distribution $q(j)$ represents instead how the agent would gamble on outcomes of the ground measurement if the measurement in the sky and the associated conditional lottery were nullified - i.e., they were to never take place at all. In the quantum case, the measurement in the sky is a SIC with $n = d^2$ outcomes; the measurement on the ground is any POVM. In pure Bayesian reasoning, there is no necessity that $q(j)$ be related to $p(i)$ and $r(j|i)$ at all. In quantum mechanics, however, there is a very specific relation, as we will see later, which contains the sum content of the Born Rule.

Starting with an initial quantum state ρ, we would assign a probability distribution $p(i)$ to the outcomes of the SIC measurement. In order to be able to say something about probabilities conditional on a particular outcome of the SIC measurement, we need to specify the post-measurement quantum state for that outcome. Here we will adopt the standard Lüders Rule (Busch, 1995, 2009), that ρ transforms to Π_i when outcome i occurs. The conditional probability for getting j in the subsequent von Neumann measurement on the ground, consequent upon i, is then precisely $r(j|i)$ defined in Eq. (57). With these assignments, Dutch-book coherence demands an assignment $s(j)$ for the outcomes on the ground that satisfies

$$s(j) = \sum_{i=1}^{d^2} p(i)r(j|i) \tag{59}$$

i.e., a probability that comes about via the Law of Total Probability, Eq. (3).

But now imagine the measurement in the sky nullifiedi.e., imagine it does not occur after all - and that the quantum system goes directly to the measurement device on the ground. Quantum mechanics tells us to make the probability assignment $q(j)$ given in Eq. (58) instead. So,

$$q(j) = (d+1)s(j) - 1 \tag{60}$$

That $q(j) \neq s(j)$ holds, regardless of the assignment of $s(j)$, is a formal expression of the idea that the "unperformed SIC had no outcomes", as explained in Sec. 2.1. But Eq. (60) tells us still more detailed information than this. It expresses a kind of "empirically extended coherence" - not implied by Dutch-book coherence alone, but formally similar to the kind of relation one gets from Dutch-book coherence. It contains a surprising amount of information about the structure of quantum mechanics.

To support this, let us try to glean some insight from Eq. (60). The most obvious thing one can note is that $\|s\rangle\!\rangle$ cannot be too sharp a probability distribution. For otherwise $q(j)$ will violate the bounds $0 \leq q(j) \leq 1$ set by Dutch-book coherence. Particularly,

$$\frac{1}{d+1} \leq s(j) \leq \frac{2}{d+1} \tag{61}$$

This in turn will have implications for the range of values possible for $p(i)$ and $r(j|i)$. Indeed if either of these distributions become too sharp (in the latter case, for too many values of i), again the bounds will be violated. This suggests that an essential part of quantum-state space structure, as expressed by its extreme points satisfying Eqs. (37) and (38), arises from the very requirement that $q(j)$ be a proper probability distribution. In the next Section, we will explore this question in greater depth.

First though, we must note the most general form of the Born Rule, when the measurement on the ground is not restricted to being of the simple von Neumann variety. So, let

$$q(j) = tr\,\rho F_j \tag{62}$$

and

$$r(j|i) = tr\,\Pi_i F_j \tag{63}$$

for some general POVM $\{F_j\}$ on the ground, with any number of outcomes, $j = 1, ..., m$. Then the Born Rule becomes

$$q(j) = (d+1)\sum_{i=1}^{d^2} p(i) r(j|i) - \frac{1}{d}\sum_{i=1}^{d^2} r(j|i) \tag{64}$$

As stated, this is the most general form of the Quantum Law of Total Probability. It has two terms, a term comprising the classical Law of Total Probability, and a term dependent only upon the sum of the conditional probabilities.

When the measurement on the ground is itself another SIC (any SIC) it reduces to

$$q(j) = (d+1)\sum_{i=1}^{d^2} p(i) r(j|i) - \frac{1}{d} \tag{65}$$

Notice the formal resemblance between this and Eq. (55) expressing unitary time evolution.

4.1 Why "Empirically Extended Coherence" Instead of Objective Quantum States?

What we are suggesting is that perhaps Eq. (64) should be taken as one of the basic axioms of quantum theory, since it provides a particularly clear way of thinking of the Born Rule as an addition to Dutch-book coherence. This addition is empirically based and gives extra normative rules, beyond the standard rules of probability theory, to guide the agent's behavior when he interacts with the physical world.

But, one may well ask, what is our problem with the standard way of expressing the Born Rule in the first place? How is introducing an addition to Dutch-book coherence conceptually any more palatable than introducing objective quantum states or objective probability distributions? For, if the program is successful, then the demand that $q(j)$ be a proper probability distribution will place necessary restrictions on $p(i)$ and $r(j|i)$. This - a skeptic would say - is *the* very sign that one is dealing with objective (or agent-independent) probabilities in the first place. Why would a personalist Bayesian accept *any* a priori restrictions on his probability assignments? And particularly, restrictions supposedly of empirical origin?

The reply is this. It is true that through an axiom like Eq. (64) one gets a restriction on the ranges of the various probabilities one can contemplate holding. But that restriction in no way diminishes the functional role of prior beliefs in the makings of an agent's particular assignments $p(i)$ and $r(j|i)$. That is, this addition to Dutch-book coherence preserves the points expressed in the quote by Keynes in Section 2 in a way that objective chance cannot.

Take the usual notion of objective chance, as given operational meaning through David Lewis' "Principal Principle" (Lewis, 1986a,b). If an event A has objective chance $ch(A) = x$, then the subjective, personalist probability an agent (any agent) should ascribe to A on the condition of knowing the chance proposition is

$$Prob\,(A|\text{“}ch(A) = x\text{”} \wedge E) = x \tag{66}$$

where E is any "admissible" proposition. There is some debate about what precisely constitutes an admissible proposition, but an example of a proposition universally accepted to be compatible in spite of these interpretive details is this:

> E = "All my experience causes me to believe A with probability 75%".

That is, upon knowing an objective chance, all prior beliefs should be overridden. Regardless of the agent's firmly held belief about A, that belief becomes irrelevant once he is apprised of the objective chance.

When it comes to quantum mechanics, philosophers of science who find something digestible in Lewis' idea, often view the Born Rule itself as a healthy serving of Principal Principle. Only, it has the quantum state ρ filling the role of chance. That is, for any agent contemplating performing a measurement $\{P_j\}$, his subjective, personal probabilities for the outcomes j should condition on knowledge of the quantum state just as one conditions with the Principal Principle:

$$Prob(j|\rho \wedge E) = tr\, \rho P_j \tag{67}$$

where E is any "admissible" proposition. Beliefs are beliefs, but quantum states are something else: They are the facts of nature that power a quantum version of the Principal Principle. In other words, in this context one has conceptually

$$\rho \to \text{``}ch(j) = tr\, \rho P_j\text{''} \tag{68}$$

But the Quantum-Bayesian view cannot sanction this. For, the essential point for a Quantum Bayesian is that there is no such thing as *the* quantum state. There are potentially as many states for a given quantum system as there are agents. And that point is not diminished by accepting the addition to Dutch-book coherence described in this paper. Indeed, it is just as with standard (non- quantum) probabilities, where their subjectivity is not diminished by normatively satisfying standard Dutch-book coherence.

The most telling reason for this arises directly from quantum statistical practice. The way one comes to a quantum-state assignment is ineliminably dependent on one's priors (Caves, 2007; Fuchs, 2002, 2009). Quantum states are not god-given, but have to be fought for via measurement, updating, calibration, computation, and any number of related activities. The only place quantum states are "given" outright - that is to say, the model on which much of the notion of an objective quantum state arises from in the first place - is in a textbook homework problem. For instance, a textbook exercise might read, "Assume a hydrogen atom in its ground state. Calculate ......". But outside the textbook it is not difficult to come up with examples where two agents looking at the same data, differing only in their prior beliefs, will asymptotically update to distinct (even orthogonal) pure quantum-state assignments for the same system (Fuchs, 2009). Here is a simple if contrived example. Consider a two-qubit system for which two agents have distinct quantum-state assignments ρ_+ and ρ_-, defined by $\rho_\pm = \frac{1}{2}(|0\rangle\langle 0|^{\otimes 2} + |\pm\rangle\langle\pm|^{\otimes 2})$ where $|\pm\rangle = 2^{-1/2}(|0\rangle \pm |1\rangle)$. These state assignments are "compatible" in several of the senses of (Brun, 2002; Caves, 2002d), yet suppose the first qubit is measured in the basis $\{|0\rangle, |1\rangle\}$ and outcome 1 is found. The two agents' post-measurement states for the second qubit are $|+\rangle$ and $|-\rangle$, respectively. See (Fuchs, 2009) for a more thorough discussion. Thus the basis for one's particular quantum-state assignment is always outside the formal apparatus of quantum mechanics. Nor, does it help to repeat over and over, as one commonly hears coming from the philosophy-of-physics community, "quantum probabilities are specified by physical law". The simple reply is, "No, they are not". The phrase has no meaning once one has taken on board that quantum states are born in probabilistic considerations, rather than being the parents of them, as laboratory practice clearly shows (Kaznady, 2009; Paris, 2004).

This is the key difference between the set of ideas be- ing developed here and the position of the objectivists: added relations for probabilities, yes, but no one of those probabilities can be objective in the sense of being any less a pure function of the agent. A way to put it more prosaically is that these normative considerations may narrow the agent from the full probability simplex to the set of quantum states, but beyond that, the formal apparatus of quantum theory gives him no guidance on which quantum state he should choose. Instead, the role of a normative reading of the Born Rule is as it is with usual Dutch book. Here is the way L. J. Savage put it rather eloquently (Savage, 1954, p. 57).

> According to the personalistic view, the role of the mathematical theory of probability is to enable the person using it to detect inconsistencies in his own real or envisaged behavior. It is also understood that, having detected an inconsistency, he will remove it. An inconsistency is typically removable in many different ways, among which the theory gives no guidance for choosing.

If an agent does not satisfy Eq. (64) with his personal probability assignments, then he is not recognizing properly the change of conditions (or perhaps we could say 'context' (we add this alternative formulation so as to place the discussion within the context of various other analyses of the idea of 'contextuality' (Appleby, 2005c; Ferrie, 2008, 2009; Grangier, 2002, 2005; Mermin, 1993; Spekkens, 2005, 2008). that a potential SIC measurement would bring about. The theory gives no guidance for which of his probabilities should be adjusted or how, but it does say that they must be adjusted or "undesirable consequences" will become unavoidable.

Expanding on this point, Bernardo and Smith put it this way in (Bernardo, 1994, p. 4):

> Bayesian Statistics offers a rationalist theory of personalistic beliefs in contexts of uncertainty, with the central aim of characterizing how an individual should act in order to avoid certain kinds of undesirable behavioral in- consistencies. . . . The goal, in effect, is to establish rules and procedures for individuals concerned with disciplined uncertainty accounting. The theory is not descriptive, in the sense of claiming to model actual behavior. Rather, it is prescriptive, in the sense of saying 'if you wish to avoid the possibility of these undesirable consequences you must act in the following way'.

So much, indeed, we imagine for the *full* formal structure of quantum mechanics (including dynamics, tensor- product structure, etc.) - that it is all or nearly all an addition to Dutch-book coherence. And specifying those "undesirable consequences" in terms independent of the present considerations is a significant part of the project of specifying the ontology underlying the Quantum- Bayesian position. But that is a goal we have to leave for future work. Let us now explore the consequences of adopting Eq. (64) as a basic statement, acting as if we do not yet know the underlying Hilbert-space structure that gave rise to it.

5 Deriving Quantum-State Space from "Empirically Extended Coherence"

Let us see how far we can go toward deriving various general features of quantum-state space from the conceptual apparatus portrayed in Figure 2. Remember that we are representing quantum states by probability vectors $\|p\rangle\!\rangle$ lying in the probability simplex Δ_{d^2}. The set of pure states is given by the solutions of either Eq. (40) or Eqs. (37) and (38), and can thus be thought of as an algebraic variety within Δ_{d^2} (Sullivant, 2006). The set of all quantum states, pure or mixed, is the convex hull of the set of pure states, i.e., the set of all convex combinations of vectors $\|p\rangle\!\rangle$ representing pure states. We want to explore how much of this structure can be recovered from the considerations summarized in Figure 2. We will have to add at least three other assumptions on the nature of quantum measurement, but at first, let us try to forget as much about quantum mechanics as we can.

Namely, start with Figure 2 but forget about quantum mechanics and forget about SICs. Simply visualize an imaginary experiment in the sky S, supplemented with various real experiments we might perform on the ground G. We postulate that the probabilities we should ascribe for the outcomes of G, are determined by the probabilities we would ascribe to the imaginary outcomes in the sky and the conditional probabilities for the outcomes of G consequent upon them, were the measurement in the sky factualized. Particularly we take Eq. (64) as a postulate:

$$q(j) = (d+1)\sum_{i=1}^{d^2} p(i)r(j|i) - \frac{1}{d} \tag{69}$$

We call this postulate the **urgleichung** (German for "primal equation") to emphasize its primary status to all our thinking. As before, $p(i)$ represents the probabilities in the sky and $q(j)$ represents the probabilities on the ground. The index i is assumed to range from 1 to d^2, for some fixed number d. The range of j will not be fixed, but in any case considered will be denoted as running from 1 to m. (For example, for some cases m might be d^2, for some cases it might be d, but it need be neither and may be something else entirely - it will depend upon which experiment we are talking about for G). We write $r(j|i)$ to represent the conditional probability for obtaining j on the ground, given that the experiment in the sky was actually performed and resulted in outcome i. When we want to suppress components, we will write vectors $\|p\rangle\!\rangle$ and $\|q\rangle\!\rangle$, and write R for the matrix with entries $r(j|i)$. By definition, R is a stochastic matrix, i.e., $\sum_j r(j|i) = 1$, but not necessarily a doubly stochastic matrix, i.e., $\sum_i r(j|i) = 1$ does not necessarily hold (Horn, 1985, pp. 526-528).

One of the main features we will require, of course, is that calculated by Eq. (69), $\|q\rangle\!\rangle$ must satisfy $0 \le q(j) \le 1$ for all j. Thus, let us also honor the special inequality

$$0 \le (d+1)\sum_{i=1}^{d^2} p(i)r(j|i) - \frac{1}{d}\sum_{i=1}^{d^2} r(j|i) \le 1 \tag{70}$$

with a name: the *fundamental inequality*.

To proceed, let us define two sets $\mathcal{P}$ and $\mathcal{R}$, the first consisting of priors for the sky $\|p\rangle\!\rangle$, and the second consisting of stochastic matrices R. The matrices R could also be regarded as part of the agent's prior, but since in this discussion we keep R fixed once the measurement on the ground is fixed, we reserve the term "prior" for members of the set $\mathcal{P}$. We shall sometimes call $\mathcal{P}$ our *state space*. and its elements *states*. We will say that $\mathcal{P}$ and $\mathcal{R}$ are *consistent* (with respect to the fundamental inequality) if

(1) for any fixed $R \in \mathcal{R}$, there is no $\|p\rangle\!\rangle \in \mathcal{P}$ that does not satisfy the fundamental inequality, and

(2) for any fixed $\|p\rangle\!\rangle \in \mathcal{P}$, there is no $R \in \mathcal{R}$ that does not satisfy the fundamental inequality.

With respect to consistent sets $\mathcal{P}$ and $\mathcal{R}$, for convenient terminology, we call a general $\|p\rangle \in \Delta_{d^2}$ *valid* if it is within the state space $\mathcal{P}$; if it is not within $\mathcal{P}$, we call it invalid.

What we want to pin down are the properties of $\mathcal{P}$ and $\mathcal{R}$ under the assumption that they are *maximal*. By this we mean that for any $\|p'\rangle\!\rangle \notin \mathcal{P}$, if we were to attempt to create a new state space $\mathcal{P}'$ by adding $\|p'\rangle\!\rangle$ to the original $\mathcal{P}$, $\mathcal{P}'$ and $\mathcal{R}$ would not be consistent. Similarly, if we were to attempt to add a new point R' to $\mathcal{R}$. In other words, when $\mathcal{P}$ and $\mathcal{R}$ are maximal, they are full up with respect to any further additions. In summary,

1. $\mathcal{P}$ and $\mathcal{R}$ are said to be *consistent* if all pairs $(\|p\rangle\!\rangle, R) \in \mathcal{P} \times \mathcal{R}$ obey the fundamental inequality.

2. $\mathcal{P}$ and $\mathcal{R}$ are said to be *maximal* whenever $\mathcal{P}' \subseteq \mathcal{P}$ and $R' \subseteq \mathcal{R}$ imply $\mathcal{P}' = \mathcal{P}$ and $\mathcal{R}' = \mathcal{R}$ for any consistent $\mathcal{P}'$ and $\mathcal{R}'$.

The key idea behind the demand for maximality is that we want the urgleichung to be as least exclusionary as possible in limiting an agent's probability assignments. There is, of course, no guarantee without further assumptions there will be a unique maximal $\mathcal{P}$ and $\mathcal{R}$ consistent with the fundamental inequality, or even whether there will be a unique set of them up to rotations or other kinds of transformations, but we can certainly say some things.

One important result follows immediately: If $\mathcal{P}$ and $\mathcal{R}$ are consistent and maximal, both sets must be convex. For instance, if $\|p\rangle\!\rangle$ and $\|p'\rangle\!\rangle$ satisfy (70) for all $R \in \mathcal{R}$ it is clear that, for any $x \in [0,1]$, $\|p''\rangle\!\rangle = x\|p\rangle\!\rangle + (1-x)\|p'\rangle\!\rangle$ will as well. Thus, if $\|p''\rangle\!\rangle$ were not in $\mathcal{P}$, the set would not have been maximal to begin with. It is important to recognize that the considerations leading to the convexity of the state space here are distinct from the arguments one finds in the "convex sets" and "operational" approaches to quantum theory. See for instance (Busch, 1995; Holevo, 1982) and more recently the BBLW school starting in (Barnum, 2006) (and several publications thereafter), as well as the work of Hardy (Hardy, 2001). There the emphasis is on the idea that a state of ignorance about a finer preparation is a preparation itself. The present argument even differs from some of our own earlier Bayesian considerations (where care was taken not to view preparation' as an objective matter of fact, independent of prior beliefs, as talk of preparation would seem to imply) (Fuchs, 2002; Schack, 2003). Here instead, the emphasis is on the closure of the fundamental inequality, i.e., maximal $\mathcal{P}$ and $\mathcal{R}$. Furthermore, maximality and the boundedness of Eq. (70) ensures that $\mathcal{P}$ and $\mathcal{R}$ are closed sets, thus convex sets with extreme points (Appleby, 2011).

Now, is there any obvious connection between $\mathcal{P}$ and $\mathcal{R}$? Let us make the innocuous assumption that one can be completely ignorant of the outcomes in the sky:

Assumption 1:
$$\|p\rangle\!\rangle = \left(\frac{1}{d^2}, \frac{1}{d^2}, \cdots, \frac{1}{d^2}\right)^T \in \mathcal{P} \tag{71}$$
Certainly for any real-world experiment, one can always be maximally ignorant of which of its outcomes will occur! Suppose now that the experiment in the sky really is performed as well as the experiment on the ground. Before either experiment, the agent is ignorant of both the outcome i in the sky and the outcome j on the ground. Using Bayes' rule, he can find the conditional probability for i given j, which has the form of a posterior probability,

$$Prob(i|j) = \frac{r(j|i)}{\sum_k r(j|k)} \tag{72}$$

Let us now make a less innocuous assumption.

Assumption 2: Principle of Reciprocity: Posteriors from Maximal Ignorance Are Priors. *For any $R \in \mathcal{R}$, a posterior probability $Prob(i|j)$ as in Eq.*

(72) is a valid prior $p(i)$ for the outcomes of the measurement in the sky. Moreover, for each valid $p(i)$, there exists some $R \in \mathcal{R}$ and some j such that $p(i) = Prob(i|j)$ as in Eq. (72).

Quantum mechanics certainly has this property. For, suppose a completely mixed state for our quantum system and a POVM $\mathcal{G} = \{G_j\}$ measured on the ground. Upon noting an outcome j on the ground, the agent will use Eqs. (63) and (72) to infer

$$Prob(i|j) = \frac{tr\,\Pi_i G_j}{d tr\,G_j} \tag{73}$$

Defining

$$\rho_j = \frac{G_j}{tr\,G_j} \tag{74}$$

this says that

$$Prob(i|j) = \frac{1}{d} tr\,\rho_j \Pi_i \tag{75}$$

In other words, $Prob(i|j)$ is itself a SIC-representation of a quantum state. Moreover, ρ_j can be any quantum state whatsoever, simply by adjusting which POVM $\mathcal{G}$ is under consideration.

5.1 Basis Distributions

Since we are free to contemplate any measurement on the ground, let us consider the case where the ground measurement is set to be the same as that of the sky. We will denote $r(j|i)$ by $r_S(j|i)$ in this special case. Remembering that the probabilities on the ground, $q(j)$, refer to the case that the measurement in the sky remains counterfactual, we must then have that $p(j) = q(j)$ for any valid $\|p\rangle\!\rangle$, or, using the urgleichung,

$$p(j) = (d+1)\sum_i p(i) r_S(j|i) - \frac{1}{d}\sum_i r_S(j|i) \tag{76}$$

Take the case where $p(i) = \frac{1}{d^2}$ specifically. Substituting for $p(i)$ in Eq. (76), we find that the $r_S(j|i)$ must satisfy

$$\sum_i r_S(j|i) = 1 \tag{77}$$

Therefore, when going back to more general priors $\|p\rangle\!\rangle$, one has in fact the simpler relation

$$p(j) = (d+1)\sum_i p(i)r_S(j|i) - \frac{1}{d} \tag{78}$$

Introducing an appropriately sized matrix M of the form

$$M = \begin{pmatrix} (d+1) - \frac{1}{d} & -\frac{1}{d} & \cdots & -\frac{1}{d} \\ -\frac{1}{d} & (d+1) - \frac{1}{d} & \cdots & -\frac{1}{d} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{d} & -\frac{1}{d} & \cdots & (d+1) - \frac{1}{d} \end{pmatrix} \tag{79}$$

we can rewrite Eq. (78) in vector form,

$$MR_S\|p\rangle\!\rangle = \|p\rangle\!\rangle \tag{80}$$

At this point, we pause for a minor assumption on our state space:

Assumption 3: *The elements $\|p\rangle\!\rangle \in \mathcal{P}$ span the full simplex Δ_{d^2}.*

This is a very natural assumption: For if $\mathcal{P}$ did not span the simplex, one would be justified in simply using a smaller simplex for all considerations.

With Assumption 3, the only way Eq. (80) can be satisfied is if

$$MR_S = I \tag{81}$$

Since M is a circulant matrix, its inverse is a circulant matrix(where each row vector is rotated one element to the right relative to the preceding row vector) as well, and one can easily work out that,

$$r_S(j|i) = \frac{1}{d+1}\left(\delta_{ij} + \frac{1}{d}\right) \tag{82}$$

It follows by the Principle of Reciprocity (our Assumption 2) then that among the distributions in $\mathcal{P}$, along with the uniform distribution, there are at least d^2 other ones, namely:

$$\|e_k\rangle\!\rangle = \left(\frac{1}{d(d+1)}, \ldots, \frac{1}{d}, \cdots, \frac{1}{d(d+1)}\right)^T \tag{83}$$

with a $\frac{1}{d}$ in the k^{th} slots and $\frac{1}{d(d+1)}$ in all other slots. We shall call these d^2 special distributions, appropriately enough, the *basis distributions*.

Notice that, in the special case of quantum mechanics, the basis distributions are just the SIC states themselves, now justified in a more general setting. Also, like the SIC states, we will have

$$\sum_i e_k(i)^2 = \frac{2}{d(d+1)} \quad \forall k \tag{84}$$

in accordance with Eq. (37).

5.2 A Bloch Sphere

Consider a class of measurements for the ground that have a property we shall call *in-step unpredictability*, ISU. The property is this: Whenever one assigns a uniform distribution for the measurement in the sky, one also as- signs a uniform distribution for the measurement on the ground. This is meant to express the idea that the measurement on the ground has no in-built bias with respect to one's expectations of the sky: Complete ignorance of the outcomes of one translates into complete ignorance of the outcomes of the other. (In the full-blown quantum mechanical setting, this corresponds to a POVM $\{G_j\}$ such that $tr\, G_j$ is a constant value - von Neumann measurements with d outcomes being one special case of this.)

Denote the $r(j|i)$ and corresponding matrix R in this special case by $r_{ISU}(j|i)$ and R_{ISU}, respectively, and suppose the measurement being spoken of has m outcomes. Our demand is that

$$\frac{1}{m} = \frac{(d+1)}{d^2} \sum_i r_{ISU}(j|i) - \frac{1}{d} \sum_i r_{ISU}(j|i) \tag{85}$$

To meet this, we must have

$$\sum_i r_{ISU}(j|i) = \frac{d^2}{m} \tag{86}$$

and the urgleichung becomes

$$q(j) = (d+1) \sum_i p(i) r_{ISU}(j|i) - \frac{d}{m} \tag{87}$$

Suppose now that a prior $\|s\rangle\!\rangle$ for the sky happens to arise in accordance with Eq. (72) for one of these ISU measurements. That is,

$$s(i) = \frac{r_{ISU}(j|i)}{\sum_k r_{ISU}(j|k)} \tag{88}$$

for some R_{ISU} and some j. The Eq. (87) tells us that for any $\|p\rangle\!\rangle \in \mathcal{P}$, we must have

$$0 \leq \frac{d^2}{m}(d+1)\sum_i p(i)s(i) - \frac{d}{m} \leq 1 \tag{89}$$

In other words, for any $\|s\rangle\!\rangle$ of our specified variety and any $\|p\rangle\!\rangle \in \mathcal{P}$, we must have

$$\frac{1}{d(d+1)} \leq \sum_i p(i)s(i) \leq \frac{d+m}{d^2(d+1)} \tag{90}$$

Think particularly on the case where $\|s\rangle\!\rangle = \|p\rangle\!\rangle$. Then we must have

$$\sum_i p(i)^2 \leq \frac{d+m}{d^2(d+1)} \tag{91}$$

Note how this compares to Eq. (84).

Now, suppose there are ISU measurements (distinct from simply bringing the sky measurement down to the ground) that have the basis distributions $\|e_k\rangle\!\rangle$ as their posteriors in the way of Assumption 2, the Principle of Reciprocity. If this is so, then note that according to Eq. (84) the bound in Eq. (91) will be violated unless $m \geq d$. Moreover, it will not be tight for the basis states unless $m = d$ precisely.

Thinking of a basis distribution as the prototype of an extreme-point state (for after all, they give the most predictability possible for the measurement in the sky), this motivates the next assumption - this one being significantly stronger than the previous two:

Assumption 4: *Every extreme point $\|p\rangle\!\rangle \in \mathcal{P}$ arises in the manner of Eq. (88) as the posterior of an ISU measurement with $m = d$ and achieves equality in Eq. (91).*

Thus, for any two extreme points $\|p\rangle\!\rangle$ and $\|s\rangle\!\rangle$, we are assuming

$$\frac{1}{d(d+1)} \leq \sum_i p(i)s(i) \leq \frac{2}{d(d+1)} \tag{92}$$

with equality in the right-hand side when $\|p\rangle\!\rangle = \|s\rangle\!\rangle$. By the way, it should be noted that this inequality establishes that if $\mathcal{P}$ at least contains the actual quantum-state space, it can contain no more than that. That is, the full set of quantum states is, in fact, a maximal set. For suppose a SIC exists, yet $\|s\rangle\!\rangle$ corresponds to some non-positive-semidefinite operator via the mapping in Eq. (30). Then there will be some $\|s\rangle\!\rangle \in \mathcal{P}$ corresponding to a pure quantum state such that the left-hand side of Eq. (92) is violated. This follows immediately from the definition of positive semi-definiteness and the expression for Hilbert-Schmidt inner products in Eq. (49).

Thus, the extreme points of $\mathcal{P}$ live on a sphere

$$\sum_i p(i)^2 = \frac{2}{d(d+1)} \tag{93}$$

Further trivial aspects of quantum-state space follow immediately from the requirement of Eq. (92) for any two extreme points. For instance, since the basis distributions are among the set of valid states, for any other valid state $\|p\rangle\!\rangle$ no component in it can be too large. This follows because

$$\langle\!\langle p\|e_k\rangle\!\rangle = \frac{1}{d(d+1)} + \frac{1}{d+1}p(k) \tag{94}$$

The right-hand side of Eq. (92) then requires

$$p(k) \le \frac{1}{d} \tag{95}$$

But, do we have enough to get to us all the way to Eq. (41) in addition to Eq. (37)? We will analyze aspects of this in the next subsection. First however, let us linger a bit over the significance of the sphere.

What we have postulated in a natural way is that the extreme points of $\mathcal{P}$ must live on a (d^2-1)-sphere centered at the zero vector. But then it comes for free that these extreme points must also live on a smaller-radius (d^2-2)-sphere centered at

$$\|c\rangle\!\rangle = \left(\frac{1}{d^2}, \frac{1}{d^2}, \dots, \frac{1}{d^2}\right)^T \tag{96}$$

This is because the $\|p\rangle\!\rangle$ live on the probability simplex Δ_{d^2}. For, let $\|w\rangle\!\rangle = \|p\rangle\!\rangle - \|c\rangle\!\rangle$, where $\|p\rangle\!\rangle$ is any point satisfying Eq. (93). Then

$$r^2 = \langle\!\langle w\|w\rangle\!\rangle = \frac{d-1}{d^2(d+1)} \tag{97}$$

gives the radius of the lower-dimensional sphere.

The sphere in Eq. (97) is actually the more natural sphere for us to think about, as most of the sphere in Eq. (93) - all but a set of measure zero - is thrown away anyway. In fact, it may legitimately be considered the higher-dimensional analog of the Bloch sphere from the Quantum-Bayesian point of view. Indeed, when $d = 2$, we have a 2-sphere, and it is isomorphic to the usual Bloch sphere.

It is natural to think of the statement

$$\sum_i p(i)^2 \leq \frac{2}{d(d+1)} \quad \text{for all } \|p\rangle\!\rangle \in \mathcal{P} \tag{98}$$

in information theoretic terms. This is because two well-known measures of the uncertainty associated with a probability assignment - the Renyi and Darćozy entropies (Acźel, 1975) of order 2 - are simple functions of the left-hand side of it. Recall the Renyi entropies most generally (defined for all $\alpha \geq 1$)

$$R_\alpha(\|p\rangle\!\rangle) = \frac{1}{1-\alpha} \ln\left(\sum_i p(i)^\alpha\right) \tag{99}$$

as well as the Darćozy entropies

$$D_\alpha(\|p\rangle\!\rangle) = \frac{1}{2^{1-\alpha}-1}\left(\sum_i p(i)^\alpha - 1\right) \tag{100}$$

In the limit $\alpha \to 1$, these both converge to the Shannon entropy. The characterization of quantum-state space appears to provide an application of these entropies for the value $\alpha = 2$.

To put it in a slogan (Caves, 1996; Fuchs, 2010), "In quantum mechanics, maximal information is not complete and cannot be completed". The sharpest pre- dictability one can have for the outcomes of a SIC measurement is specified by Eq. (37). This is an old idea, of course, but quantified here in yet another way. It is related to the basic idea underlying the toy model of R. W. Spekkens (Spekkens, 2007), with its "knowledge balance principle". In that model, which combines local hidden variables with an "epistemic constraint" on the agent's knowledge of the variables' values, more than twenty well-known quantum information theoretic phenomena (like no-cloning (Dieks, 1982; Wootters, 1982), no-broadcasting (Barnum, 1996), teleportation (Bennett, 1993), correlation monogamy (Coffman, 2000), "nonlocality without entanglement" (Bennett, 1999), etc.) are readily reproduced, at least in a qualitative way.

Despite the toy model's impressive successes, however, we suspect that an information constraint alone cannot support the more sweeping part of the Quantum Bayesian program, that "the possible outcomes cannot correspond to actualities, existing objectively prior to asking the question", i.e., that unperformed measurements have no outcomes. In ways, there is a world of difference be- tween the present considerations to do with an addition to Dutch-book coherence and "epistemic restriction" approaches. First, it is hard to see how that line of thought can get beyond the possibility of an underlying hidden-variable model (as the toy model illustrates). Second, and more importantly, in the present approach the Bloch sphere may well express an epistemic constraint - a constraint on an agent's advised certainty. But the epistemic constraint is itself a result of a deeper consideration to do with the coherence between factual and counterfactual gambles, not a starting point. Furthermore, the constraint is not expressible in terms of a single information function anyway; instead it involves pairs of distributions. We go on to explain this point.

5.3 But Only Part of It

The state-space implied by Eq. (92) does not lead to the full sphere in Eq. (97). According to the left-hand side of Eq. (92), when two points are too far away from each other, at least one of them cannot be in $\mathcal{P}$. We will show this more carefully in the next section: that the extreme $\|p\rangle\!\rangle \in \mathcal{P}$ comprise only part of a sphere. Of some interest, however, is that Eq. (97) already tells us that we cannot have the full sphere as well. For, the radius of the sphere is such that the sphere extends beyond the boundary of the probability simplex Δ_{d^2}. Hence. $\mathcal{P}$ is contained within a nontrivial intersection of sphere and simplex.

This is established by a nice argument due to Gabriel Plunk (Plunk, 2002). Let us calculate the shortest distance between $\|c\rangle\!\rangle$ and an n-flat of the simplex - an n-flat is defined so that it contains only probability vectors with n

vanishing components. For instance, all $\|p\rangle\!\rangle$ of the form

$$\|p\rangle\!\rangle = \left(p(1), p(2), ..., p(d^2 - n), 0, 0, ..., 0\right)^T \tag{101}$$

with $d^2 - n$ initial nonvanishing components and n final vanishing components, form an n-flat. A more general n-flat would have all the vanishing and nonvanishing components interspersed.

What is the minimal distance $D_{min}(\|c\rangle\!\rangle, \|p\rangle\!\rangle)$ between the center point and an n-flat? Taking

$$\begin{aligned} D^2(\|c\rangle\!\rangle, \|p\rangle\!\rangle) = & \sum_{i=1}^{d^2-n} \left(p_n(i) - \frac{1}{d^2}\right)^2 \\ & + \sum_{i=d^2-n+1}^{d^2} \left(0 - \frac{1}{d^2}\right)^2 \end{aligned} \tag{102}$$

generally, and recognizing the constraint

$$\sum_{i=1}^{d^2-n} p_n(i) = 1 \tag{103}$$

we can use the calculus of variations to find

$$D_{min}^2(\|c\rangle\!\rangle, \|p\rangle\!\rangle) = \frac{n}{d^2(d^2 - n)} \tag{104}$$

Can there be an n for which

$$D_{min}^2(\|c\rangle\!\rangle, \|p\rangle\!\rangle) < r^2 \ \ ? \tag{105}$$

where r is defined by Eq. (97)? In other words, can the sphere ever poke outside of the probability simplex? Just solving the inequality for n gives $n < \frac{1}{2}d(d-1)$.

Thus, when $n < \frac{1}{2}d(d-1)$, the point

$$\|p_{s(n)}\rangle\!\rangle = \left(\frac{1}{d^2 - n}, \frac{1}{d^2 - n}, ..., \frac{1}{d^2 - n}, 0, 0, ..., 0\right)^T \tag{106}$$

on an n-flat surface of the simplex lies within the sphere the extreme points of $\mathcal{P}$ inhabit. Only in the case of the qubit, $d = 2$, does the sphere reside completely within the simplex - the set is equivalent to the well-known Bloch sphere.

A corollary to Plunk's derivation is that we can put a (weak) bound on the maximum number of zero components a valid $\|p\rangle\rangle$ can contain. To have n zero components, $\|p\rangle\rangle$ must live on an n-flat. But extreme $\|p\rangle\rangle$ are always a distance $D^2_{extreme} = \frac{d-1}{d^2(d+1)}$ from $\|c\rangle\rangle$. So, if n is such that

$$D^2_{min}\left(\|c\rangle\rangle, \|p\rangle\rangle\right) > D^2_{extreme} \tag{107}$$

then $\|p\rangle\rangle$ cannot live on an n-flat. This limits n: if $n > \frac{1}{2}d(d-1)$, then a state cannot live on that n-flat. Thus, for a valid $\|p\rangle\rangle$, there is an upper bound to how many zero components it can have

$$n_{zeros} \leq \frac{1}{2}d(d-1) \tag{108}$$

Our first inclination was that this is surely a weak bound. But even in quantum mechanics, we know of no better bound than this. This follows from the best bound we are aware of in that context, a Hilbert-space bound of Delsarte, Goethels, and Seidel (Delsarte, 1975) (which we note can also be proven by elementary Gram matrix methods in Hilbert-Schmidt space). Let $P_i, i = 1, ..., v$, be a set of rank-1 projection operators on an f-dimensional Hilbert space $\mathcal{H}_f$ such that $tr\, P_i P_j = c$, for all $i \neq j$. then

$$v \leq \frac{f(1-c)}{1-fc}$$

To find the maximum number of zero components $\|p\rangle\rangle$ can contain, we just need to ask the question of how many SIC vectors can possibly fit in a $(d-1)$-dimensional subspace. Inserting the parameters $f = d-1$ and $c = 1/(d+1)$ into this bound, we find $n_{zeros} \leq \frac{1}{2}d(d-1)$. Interestingly, this bound is saturated when $d = 2$ and $d = 3$. On the other hand, in dimensions $d = 4$ and $d = 5$, D. M. Appleby has checked exhaustively for the known SICs that never more than $d-1$ of the vectors fit within a $(d-1)$–dimensional subspace (Appleby, 2012, private communication).

However, an alternative and more direct argument for Eq. (108) is this - it is a straightforward application of the Schwarz inequality:

$$
\begin{aligned}
1 &= \left(\sum_{nonzero\,terms} p(i) \right)^2 \\
&\leq (d^2 - n_{zeros}) \left(\sum_{nonzero\,terms} p(i)^2 \right) \\
&= (d^2 - n_{zeros}) \frac{2}{d(d+1)}
\end{aligned}
\tag{109}
$$

Eq. (108) follows immediately.

But this is only the beginning of the trimming of the Bloch sphere: More drastic restrictions come from the left-hand of the fundamental inequality.

5.4 An Underlying 'Dimensionality'?

What else does the inequality in Eq. (92) imply? Here is at least one more low hanging fruit. The left side of Eq. (92) signifies that the "most orthogonal" two valid distributions $\|p\rangle\!\rangle$ and $\|q\rangle\!\rangle$ can ever be is

$$\langle\!\langle p\|q\rangle\!\rangle = \sum_i p(i)q(i) = \frac{1}{d(d+1)} \tag{110}$$

Their overlap can never approach zero; they can never be truly orthogonal. Now, suppose we have a collection of distributions $\|p_k\rangle\!\rangle, k = 1, ..., n$, all of which live on the sphere - that is, they individually saturate the right-hand side of Eq. (92). We can ask, how large can the number n can be while maintaining that each of the $\|p_k\rangle\!\rangle$ be maximally orthogonal to each other. Another way to put it is, what is the maximum number of "mutually maximally distant" states?

In other words, we would like to satisfy

$$\langle\!\langle p_k\|p_l\rangle\!\rangle = \frac{\delta_{kl} + 1}{d(d+1)} \tag{111}$$

for as many values as possible. It turns out that there is a nontrivial constraint on how large n can be, and it is nothing other than $n = d$ - the same thing one sees in quantum mechanics.

To see this, let us again reference the center of the probability simplex with all our vectors. Define

$$\|w_k\rangle\!\rangle = \|p_k\rangle\!\rangle - \|c\rangle\!\rangle \tag{112}$$

In these terms, our constraint becomes

$$\langle\!\langle w_k\|w_l\rangle\!\rangle = \frac{d\delta_{kl} - 1}{d^2(d+1)} \tag{113}$$

However, notice what this means: We are asking for a ?set of vectors whose Gram matrix $G = [\langle\!\langle w_k\|w_l\rangle\!\rangle]$ is an $n \times n$ matrix of the form

$$G = \begin{pmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a \end{pmatrix} \tag{114}$$

with

$$a = \frac{d-1}{d^2(d+1)} \quad \text{and} \quad b = \frac{-1}{d^2(d+1)} \tag{115}$$

By an elementary theorem in linear algebra, a *proposed* set of vectors with a *proposed* Gram matrix G can exist if and only if G is positive semi-definite (Horn, 1985, pp. 407-408). Moreover, the rank of G represents the number of linearly independent such vectors. (We write "proposed" because if G is not positive semi-definite, then of course there are no such vectors).

Since G in Eq. (114) is a circulant matrix, its eigenvalues can be readily calculated: one takes the value

$$\lambda_0 = a + (n-1)b = \frac{d-n}{d^2(d+1)} \tag{116}$$

while all the $n-1$ others are

$$\lambda_k = a - b = \frac{1}{d(d+1)} \tag{117}$$

To make G positive semi-definite, then, we must have $n \leq d$, with $n = d$ being the maximal value. At that point G is a rank-$(d-1)$ matrix, so that only $d-1$ of the $\|w_l\rangle\!\rangle$ are linearly independent.

On the other hand, all d vectors $\|w_k\rangle\!\rangle = \|p_k\rangle\!\rangle - \|c\rangle\!\rangle$ actually are linearly independent. To see this, suppose there are numbers α_i such that $\sum_i \alpha_i \| p_i \rangle\!\rangle = 0$. Acting from the left on this equation with $\langle\!\langle c \|$, one obtains

$$\sum_i \alpha_i = 0 \tag{118}$$

On the other hand, acting on it with $\langle\!\langle p_k \|$, we obtain

$$\begin{aligned} 0 &= \frac{2}{d(d+1)}\alpha_k + \frac{1}{d(d+1)}\sum_{i \neq k}\alpha_i \\ &= \frac{1}{d(d+1)}\alpha_k + \frac{1}{d(d+1)}\sum_i \alpha_i \\ &= \frac{1}{d(d+1)}\alpha_k \end{aligned} \tag{119}$$

So, indeed,

$$\sum_i \alpha_i \| p_i \rangle\!\rangle = 0 \quad \Rightarrow \quad \alpha_k = 0 \;\; \forall k \tag{120}$$

What this reveals is a significantly smaller "dimension" for the valid states on the surface of the sphere than one might have thought. A priori, one might have thought that one could get nearly d^2 maximally equidistant points on the sphere, but it is not so - only d instead. This is certainly a suggestive result, but "dimension" at this stage must remain in quotes. Ultimately one must see that the Hausdorff dimension of the manifold of valid extreme states is $2d - 2$ (i.e., what it is in quantum theory), and the present result does not get that far.

5.5 Summary of the Argument So Far

Let us summarize the assumptions made to this point and summarize their consequences as well.

Assumption 0: The Urgleichung. *See Figure 2. De- grees of belief for outcomes in the sky and degrees of belief for outcomes on the ground ought to be related by this fundamental equation:*

$$q(j) = (d+1)\sum_{i=1}^{d^2} p(i)r(j|i) - \frac{1}{d}\sum_{i=1}^{d^2} r(j|i) \tag{121}$$

From the urgleichung, the fundamental inequality arises by the requirement that $0 \leq q(j) \leq 1$ always. The sets $\mathcal{P}$ and $\mathcal{R}$ are defined to be sets of priors

$\|p\rangle\rangle$ and stochastic matrices R, that are consistent and maximal.

Assumption 1: The state of maximal ignorance $\|c\rangle\rangle$ in Eq. (96) is included in $\mathcal{P}$.

Assumption 2: Principle of Reciprocity: Posteriors from Maximal Ignorance Are Priors. *For any $R \in \mathcal{R}$, a posterior probability consequent upon outcome j of a ground measurement,*

$$Prob(i|j) = \frac{r(j|i)}{\sum_k r(j|k)} \tag{122}$$

may be taken as a valid prior $p(i)$ for the outcomes of the measurement in the sky. Moreover, all valid priors $p(i)$ may arise in this way.

Assumption 3: $\mathcal{P}$ spans the probability simplex Δ_{d^2}.

Assumption 4: Extreme-Point Preparations. *The extreme points of the convex set $\mathcal{P}$ may all be generated as the posteriors of a suitably chosen ground measurement for which maximal ignorance of sky outcomes implies maximal ignorance of ground outcomes. Moreover, these measurements all have the minimum number of outcomes consistent with generating the basis distributions $\|e_k\rangle\rangle$ in this way.*

With these four assumptions, we derived that the basis distributions $\|e_k\rangle\rangle$ should be among the valid states $\mathcal{P}$. We derived that for any $\|p\rangle\rangle \in \mathcal{P}$, the probabilities are bounded above by $p(k) < \frac{1}{d}$. We derived that the extreme points of the valid $\|p\rangle\rangle$ should live on the surface of a sphere that at times pokes outside the probability simplex. We found a bound, given in Eq. (108), on the number of zero components of $\|p\rangle\rangle$ that is as good as the best known bound that has been derived using the conventional quantum formalism. Most particularly we derived that for any two valid distributions $\|p\rangle\rangle$ and $\|s\rangle\rangle$ (including the case where $\|p\rangle\rangle = \|s\rangle\rangle$), it must hold that

$$\frac{1}{d(d+1)} \leq \sum_i p(i)s(i) \leq \frac{2}{d(d+1)} \tag{123}$$

From the latter, it follows that no more than d extreme points $\|p\rangle\rangle$ can ever be mutually maximally distant from each other. Furthermore we showed that not every flat zeros-bound vector can be a valid $\|p\rangle\rangle$.

These are all hints that our structure might just be isomorphic to quantum-state space under the assumption that SICs exist. What really needs to be derived is that the extreme points of such a convex set correspond to an algebraic variety of the form

$$p(k) = \alpha \sum_{i=1}^{n} p(i)r(j|i) - \beta \sum_{i=1}^{n} r(j|i) \tag{124}$$

as given in Eq. (40), with a set of c_{ink} that can be writ- ten in the form of Eq. (39). Whether this step can be made without making any further assumptions, we do not know. Nor do we have a strong feeling presently of whether the auxiliary Assumptions 1-4 are the ones best posited for achieving our goal. The key idea is to supplement Assumption 0 with as little extra structure as possible for getting all the way to full-blown quantum mechanics. Much work remains, both at the technical and conceptual level.

6 Relaxing the Constants and Regaining Them

But what is the origin of the urgleichung in the first place? In this Section, which in part follows closely (Fuchs, 2011), we take a small step toward a deeper understanding of the particular form our relation takes in Eq. (121). We do this by initially generalizing away from Eq. (121) and then testing what it takes to get back to it. What we mean by this is that we should imagine the more general set-up in Figure 2, where the number of outcomes for the measurement in the sky is potentially some more general number n (not initially assumed to be a perfect square d^2). Furthermore we drop away all traces of the parameter d, by considering a generalized urgleichung with two initially arbitrary parameters α and β. That is to say, for this Section, our fundamental postulate will be:

Resorption 0: Generalized Urgleichung. *For whichever experiment we are talking about for the ground, $q(j)$ should be calculated according to*

$$q(j) = \alpha \sum_{i=1}^{n} p(i)r(j|i) - \beta \sum_{i=1}^{n} r(j|i) \tag{125}$$

where α and β re fixed nonnegative real numbers.

Otherwise, all considerations will be the same as they were in the beginning of Section 5. Particularly, the measurements on the ground can have any number m of outcomes, where the value m in any individual case will be set by the details of the measurement under consideration at that time. Our goal will be to see what assumptions can be added to this basic scenario so that the urgleichung in Eq. (121) re-arises in a natural manner. That is to say, we would like to see what assumptions can be added to this recipe so that $\alpha = d+1$, $\beta = \frac{1}{d}$, and $n = d^2$ (for some d) are the end result.

Immediately, one can see that n,α, and β cannot be independent. This just follows from the requirements that

$$\sum_{j=1}^{m} q(j) = 1 \quad , \quad \sum_{m=1}^{m} r(j|i) = 1 \;\; \forall i \quad \text{and} \quad \sum_{i=1}^{n} p(i) = 1 \tag{126}$$

Summing both left and right sides of Eq. (125) over j, one obtains,

$$n\beta = \alpha - 1 \tag{127}$$

Furthermore, since $\beta \neq 0$ is assumed, requiring $q(j) \geq 0$ necessitates

$$\frac{\alpha}{\beta} \geq \frac{\sum_i r(j|i)}{\sum_i p(i) r(j|i)} \geq 1 \tag{128}$$

As before, we now start studying the consequences of the full requirement that $0 \leq q(j) \leq 1$, in the form of a generalized fundamental inequality:

$$0 \leq \alpha \sum_{i=1}^{n} p(i) r(j|i) - \beta \sum_{i=1}^{n} r(j|i) \leq 1 \tag{129}$$

The two sets $\mathcal{P}$ and $\mathcal{R}$ are defined analogously to the discussion just after Eq. (70), the first a set of priors for the sky and the second a set of conditionals for the ground (given the outcomes i in the sky). $\mathcal{P}$ and $\mathcal{R}$ are taken to be consistent and maximal.

Two assumptions, we shall borrow straight away from our previous development in Section 5.

Resumption 1: Principle of Reciprocity: Posteriors from Maximal Ignorance Are Priors. *For any $R \in \mathcal{R}$, a posterior probability, consequent upon outcome j of a ground measurement,*

$$Prob(i|j) = \frac{r(j|i)}{\sum_k r(j|k)} \tag{130}$$

may be taken as a valid prior $p(i)$ for the outcomes of the measurement in the sky. Moreover, all valid priors $p(i)$ may arise in this way.

Resumption 2: Basis states span the simplex Δ_{d^2}. *The conditional probabilities $r(j|i)$ derived from setting the ground measurement equal to the sky measurement give rise to posterior distributions $\|e_k\rangle\!\rangle$, via Eq. (130), that span the whole probability simplex.*

At this stage, the argument goes just as it did in Section 5.1. In terms of the constants α and β, the components $e_k(i)$ of the basis states satisfy the equations

$$e_k(i) = \frac{1}{\alpha}(\delta_{ki} + \beta) \tag{131}$$

and

$$\sum_i e_k(i)^2 = \frac{1}{\alpha^2}\left(1 + 2\beta + n\beta^2\right) \tag{132}$$

Let us now consider a measurement with in-step unpredictability for a measurement on the ground with m outcomes ($m \neq n$) - that is, a measurement on the ground such that if one has a flat distribution for the outcomes in the sky, one will also have a flat distribution for the outcomes on the ground. Let us again denote $r(j|i)$ by $r_{ISU}(j|i)$ in this special case. Following the manipulations we did before, we must have

$$\sum_i r_{ISU}(j|i) = \frac{n}{m} \tag{133}$$

By the Principle of Reciprocity, this ISU measurement gives rise to a class of priors which we denote by $\|p_k\rangle\!\rangle, k = 1, ..., m$. Their components are given by

$$p_k(i) = \frac{m}{n} r_{ISU}(k|i) \tag{134}$$

each vector $\|p_k\rangle\!\rangle$ represents a valid prior in the sky.

Let us now introduce a new notion that we did not make use of in the previous development: We shall say that a measurement with in-step unpredictability achieves the ideal of certainty if $\|p\rangle\!\rangle = \|\|p_k\rangle\!\rangle$ implies that $q(j) = \delta_{jk}$, i.e., for such a measurement and a prior in the sky given by $\|\|p_k\rangle\!\rangle$, the agent is certain that the outcome on the ground will be k.

This leads to the following assumption:

Resumption 3: Availability of Certainty. *or any system, there is a measurement with in-step unpredictability of some number $m_0 \geq 2$ of outcomes that (i) achieves the ideal of certainty and (ii) for which one of the priors $\|p_k\rangle\rangle$ defined in Eq. (134) has the form of a basis distribution (131).*

In several axiomatic developments of quantum theory - see for instance (Goyal, 2008) and (Hardy, 2001) - the idea of repeated measurements giving rise to certainty (and the associated idea of "distinguishable states") is viewed as fundamental. However, from the Quantum-Bayesian view where *all* measurements are generative of their outcomes - i.e., outcomes never preexist the act of measurement - and certainty is always subjective certainty (Caves, 2007), the consistency of adopting a state of certainty as one's state of belief, even in what is judged to be a repeated experiment, is not self-evident at all. In fact, from this point of view, why one ever has certainty is the greater of the mysteries.

For a measurement of this type, we have

$$\langle\langle p_j \| p_k \rangle\rangle = \frac{1}{alpha}\left(\frac{m_0}{n}\delta_{jk} + \beta\right) \quad j,k = 1, ..., m_0 \tag{135}$$

where $\langle\langle \cdot \| \cdot \rangle\rangle$ denotes the inner product. Using condition (ii) of the above resumption, it follows that the squared norm $\langle\langle p_k \| p_k \rangle\rangle$ of any of the vectors $\|p_k\rangle\rangle$ is equal to the squared norm of the basis vectors given by Eq. (132). This, together with Eq. (127) now implies that

$$\frac{m_0}{n}\alpha - \beta = 1 \tag{136}$$

for any measurement satisfying Resumption 3.

Equation (135) expresses that any two of the vectors $\|p_k\rangle\rangle$ differ by the same angle, θ, defined by

$$\cos\theta = \frac{\langle\langle p_1 \| p_2 \rangle\rangle}{\langle\langle p_1 \| p_1 \rangle\rangle} \tag{137}$$

Using the relations (127) and (136) between our four variables, α, β, n and m_0 established above, this angle can be seen to equal

$$\cos\theta = \frac{n - m_0}{(m_0 - 1)^2 + n - 1} \tag{138}$$

We are now ready to state our last resumption.

Resumption 4: Many Systems, Universal Angle. *The identity of a system is parameterized by its pair* (n, m_0). *Nonetheless for all systems, the angle* θ *between pairs of priors* $\|p + k\rangle\!\rangle$ *for any measurement satisfying Resumption 3 is a universal constant given by* $\cos\theta = 1/2$.

The value $\cos\theta = 1/2$ is less arbitrary than it may appear at first sight. Taken by itself, the assumption that θ is universal implies that, for any $m_0 \geq 2$, there is an integer n such that the right-hand side of Eq. (138) evaluates to the constant $\cos\theta$. It is not hard to show that this is possible only if this constant is of the form

$$\cos\theta = \frac{q}{q+2} \tag{139}$$

where q is a non-negative integer. The universal angle postulated above corresponds to the choice $q = 2$.

Every choice for q leads to a different relation between n and m_0. For $q = 0$, we find $n = m_0$, in which case the urgleichung turns out to be identical to the classical law of total probability. For $q = 1$, we get the relationship $n = \frac{1}{2}m_0(m_0 + 1)$ which, although this fact plays no role in our argument, is characteristic of theories defined in real Hilbert space (Wootters, 1986). And for $q = 2$, we obtain

$$n = m_0^2 \tag{140}$$

Equations (136) and (140) hold for the special mea- surement postulated in Resumption 3. If we eliminate m_0 from these equations we find, with the help of Eq. (127), the relationships

$$n = (\alpha - 1)^2 \quad , \quad \beta = \frac{1}{\sqrt{n}} \tag{141}$$

These equalities must hold for *any* measurement on the ground. If we denote the integer $\alpha - 1$ by the letter d, we recover the constants of the original urgleichung of Eq. (121).

Let us reiterate slightly the philosophy here. The numerical relations between the constants α, β, and n, and in particular the fact that n is a perfect square, follow from the existence of a single special measurement de- fined in

Resumption 3, together with the postulate of a universal angle in Resumption 4. These last two resumptions, as well as the first three, are given purely in terms of the personalist probabilities a Bayesian agent may as- sign to the outcomes of certain experiments. Nowhere in all this do we mention amplitudes, Hilbert space, or any other part of the usual apparatus of quantum mechanics.

7 Summary: From Quantum Interference to Quantum Bayesian Coherence

In this discussion, we hope to have given a new and useful way to think of quantum interference: Particularly, we have shown how to view it as an empirical addition to Dutch-book coherence, operative when one calculates probabilities for the outcomes of a factualizable quantum experiment in terms of one explicitly assumed counterfactual. We did this and not once did we use the idea of a probability *amplitude*. Thus we believe we have brought the idea of quantum interference formally much closer to its root in probabilistic considerations. For this, we were aided by the mathematical machinery of SIC measurements.

In so doing we showed that the Born Rule can be viewed as a relation between probabilities, rather than a *setter* of probabilities from something more firm or secure than probability itself, i.e., rather than facilitating a probability assignment from *the* quantum state. From the Quantum-Bayesian point of view there is no such thing as *the* quantum state, there being as many quantum states for a system as there are agents interested in considering it. This last point makes it particularly clear why we needed a way of viewing the Born Rule as an extension of Dutch-book coherence: One can easily in- vent situations where two agents will update to divergent quantum states (even pure states, and even orthogonal pure states, see Footnote 19) by looking at the same empirical data (Fuchs, 2002, 2004, 2009) - a quantum state is always ultimately dependent on the agent's priors. But there is much more to do. We gave an indication that the urgleichung and considerations to do with it already specifies a significant fraction of the structure of quantum states - and for that reason one might want to take it as one of the fundamental axioms of quantum mechanics. We did not, however, get all the way back to a set based on the manifold of pure quantum states, Eq. (40). A further open question concerns the origin of the urgleichung. An intriguing idea would be to justify it Dutch-book style in terms of bought and returned lottery tickets consequent upon the nullification step in our standard scenario. Then the positive content of the Born Rule might be viewed as a kind of cost excised whenever one factualizes a SIC. But this is just speculation.

What is firm is that we have a new setting for quantifying the old idea that, in quantum mechanics, unperformed measurements have no outcomes.

8 Outlook

> Of every would be describer of the universe one has a right to ask immediately two general questions. The first is: "What are the materials of your universe's composition?" And the second: "In what manner or manners do you represent them to be connected?"
>
> \- William James, notebook entry, 1903 or 1904

This paper has focussed on adding a new girder to the developing structure of Quantum Bayesianism ('QBism' hereafter). As such, we have taken much of the previ-ously developed program as a background for the present efforts. The core arguments for why we choose a more 'personalist Bayesianism' rather than a so-called 'objective Bayesianism' can be found in (Fuchs, 2002, 2004, 2013). The argument for why a subjective, personalist account of certainty is crucial for breaking the impasse set by the EPR criterion of reality are explained in (Caves, 2007; Fuchs, 2013). Similarly for other questions on the program.

Still, fearing James' injunction, in this Section we want to discuss anew the term 'measurement', which we have been using uncritically in the present discussion. Providing a deeper understanding of the proclamation 'Unperformed measurements have no outcomes!' is, we feel, the first step toward characterizing "the materials of our universe's composition".

We take our cue from John Bell. Despite our liberal use of the term so far, we think the word 'measurement' should indeed be banished from fundamental discussions of quantum theory (Bell, 1990). For an argument in some sympathy with our own, see N. D. Mermin's "In Praise of Measurement" (Mermin, 2006). However, it is not because the word is "unprofessionally vague and ambiguous", as Bell said of it (Bell, 1987). To be sure, the are plenty of things vague and inconsistent in the writings of Bohr, Pauli, Heisenberg, von Weizsäcker, Peierls, and Peres (representatives of the so-called 'orthodoxy'), but this word we believe is not one of them. Rather, it is because, from the QBist perspective, the word suggests a misleading notion of the very subject matter of quantum mechanics.

To make the point dramatic, let us put quantum theory to the side for a moment, and consider instead basic Bayesian probability theory. There the subject matter is an agent's expectations for various outcomes. For instance, an agent might write down a joint probability distribution $P(h_i, d_j)$ for various mutually exclusive hypotheses $h_i, i = 1, ..., n$ and data values $d_j, j = 1, ..., m$, appropriate to some phenomenon. As discussed above, a major role of the theory is that it provides a scheme (Dutch-book coherence) for how these probabilities should be related to other probabilities, $P(h_i)$ and $P(d_j)$ say, as well as to any other degrees of belief the agent has for other phenomena. The theory also prescribes that if the agent is given a specific data value d_j, he should update his expectations for everything else within his interest. For instance, under the right conditions (Diaconis, 1982; Fuchs, 2012; Skyrms, 1987b), he should reassess his probabilities for the h_i by conditionalizing:

$$P_{NEW}(h_i) = \frac{P(h_i, d_j)}{P(d_j)} \tag{142}$$

But what is this phrase "given a specific data value"? What does it really mean in detail? Shouldn't one specify a mechanism or at least a chain of logical and/or physical connectives for how the raw fact signified by d_j comes into the field of the agent's consciousness? And who is this "agent" reassessing his probabilities anyway? Indeed, what is the precise definition of an agent? How would one know one when one sees one? Can a dog be an agent? Or must it be a person? Maybe it should be a person with a PhD? (Tongue-in-cheek reference to Bell again (Bell, 1990)).

Probability theory has no chance of answering these questions because they are not questions within the subject matter of the theory. Within probability theory, the notions of "agent" and "given a data value" are primitive and irreducible. Guiding agents' decisions based on data is what the whole theory is constructed for. As such, agents and data are the highest elements within the structure of probability theory - they are not to be constructed from it, but rather the former are there to receive the theory's guidance, and the latter are there to designate the world external to the agent.

QBism says that, if all of this is true of Bayesian probability theory in general, it is true of quantum theory as well. As the foundations of probability theory dismisses the questions of where data comes from and what constitutes an agent - these questions never even come to its attention - so can the foundations of quantum theory dismiss them too. This point is one of the strongest reasons for making the QBist move in the first place.

A likely reaction at this point will be along these lines: "It is one thing to say all this of probability theory, but quantum theory is a wholly different story. Quantum mechanics is no simple branch of mathematics, be it probability or statistics. Nor can it plausibly be a theory about the insignificant specks of life in our vast universe making gambles and decisions. Quantum mechanics is one of our best theories of the world! It is one of the best maps we have drawn yet of what is actually out there." But this is where QBism begs to differ. Quantum theory is *not* a 'theory of the world'. Just like probability theory is not a theory of the world, quantum theory is not as well: It is a theory for the use of agents immersed in and interacting with a world of a particular character, "the quantum world".

This last statement is crucial for understanding what we are trying to say. Regarding the idea of a world external to the agent, it must be as Martin Gardner says (Gardner, 1983),

> The hypothesis that there is an external world, not dependent on human minds, made of something, is so obviously useful and so strongly confirmed by experience down through the ages that we can say without exaggerating that it is better confirmed than any other empirical hypothesis. So useful is the posit that it is almost impossible for anyone except a madman or a professional metaphysician to comprehend a reason for doubting it.

Yet there is no implication in these words that quantum theory, for all its success in chemistry, physical astronomy, laser making, and so much else, must be read as a theory of the world. There is room for a significantly more interesting form of dependence: Quantum theory is conditioned by the character of the world, but yet is not a theory directly of it. Confusion on this very point, we believe, is what has caused most of the discomfort in quantum foundations in the 85 years since the theory's coming to a relatively stable form in 1927.

Returning to our discussion of Bell and the word "measurement", we wish the word banished because it subliminally whispers the philosophy of its birth: That quantum mechanics should be conceived in a way that makes no ultimate reference to agency, and that agents are constructed out of the theory, rather than taken as the primitive entities the theory is meant to aid. In a nutshell, the word deviously carries forward the impression that quantum mechanics should be viewed as a theory directly of the world.

Fixing the word "measurement" is the prerequisite to a new ontology - in other words, prerequisite to a statement about the (hypothesized) character of the world that does not make direct reference to our actions and gambles within it. Therefore, as a start, let us rebuild quantum mechanics in terms more conducive to the Quantum Bayesian program.

8.1 The Paulian Idea and the Jamesian Pluriverse

The best way to begin a more thoroughly QBist delineation of quantum mechanics is to start with two quotes on personalist Bayesianism itself. The first is from Hampton, Moore, and Thomas (Hampton, 1973),

> Bruno de Finetti believes there is no need to assume that the probability of some event has a uniquely determinable value. His philosophical view of probability is that it ex- presses the feeling of an individual and cannot have meaning except in relation to him.

and the second from D. V. Lindley (Lindley, 1982),

> The Bayesian, subjectivist, or coherent, paradigm is egocentric. It is a tale of one person contemplating the world and not wishing to be stupid (technically, incoherent). He realizes that to do this his statements of uncertainty must be probabilistic.

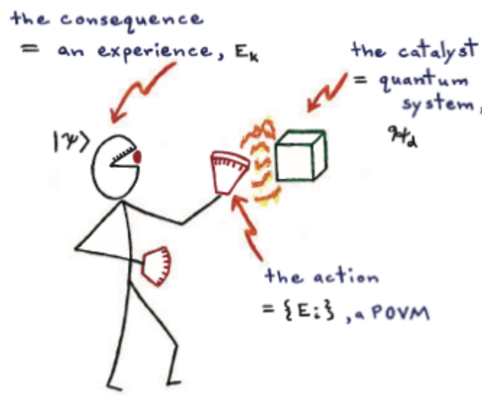

Figure 3: The Paulian Idea - in the form of a figure inspired by John Archibald Wheeler. In contemplating a quantum measurement, one makes a conceptual split in the world: one part is treated as an agent, and the other as a kind of reagent or catalyst (one that brings about change in the agent). In older terms, the former is an observer and the latter a quantum system of some finite dimension d. A quantum measurement consists first in the agent taking an *action* on the quantum system. The action is formally captured by some POVM $\{E_i\}$. The action leads generally to an incompletely predictable *consequence*, a particular personal experience E_i or the agent. The quantum state $|\psi\rangle$ makes no appearance but in the agent's head; for it only captures his degrees of belief concerning the consequences of his actions, and - in contrast to the quantum system itself - has no existence in the external world. Measurement devices are depicted as prosthetic hands to make it clear that they should be considered an integral part of the agent. (This contrasts with Bohr's view where the measurement device is always treated as a classically describable system external to the observer.) The sparks between the measurement-device hand and the quantum system represent the idea that the consequence of each quantum measurement is a unique creation within the previously existing universe. Wolfgang Pauli characterized this picture as a "wider form of the reality concept" than that of Einstein's, which he labeled "the ideal of the detached observer". What is important for modern developments is that the particular character of the catalysts - i.e., James' "materials of your universe's composition" - must leave its trace in the formal rules that allow us to conceptualize factualizable measurements in terms of a standard counterfactual one.

These two quotes make it clear that personalist Bayesianism is a "single-user theory". Thus, QBism must inherit at least this much egocentrism in its view of quantum states ρ. The "Paulian Idea" (Fuchs, 2010) - which is also essential to the QBist view - goes further still. It says that the outcomes to quantum measurements are single-user as well. That is to say, when an agent writes down her degrees of belief for the outcomes of a quantum measurement, what she is writing down are her degrees of belief about her potential *personal* experiences arising in consequence of her actions upon the external world (Fuchs, 2009, 2010b, 2012; Mermin, 2012).

Before exploring this further, let us partially formalize in a quick outline the structure of quantum mechanics from this point of view, at the moment retaining the usual mathematical formulation of the theory, but starting the process of changing the English descriptions of what the term "quantum measurement" means.

1. Primitive notions: a) the agent, b) things external to the agent, or, more commonly, "systems", c) the agent's actions on the systems, and d) the consequences of those actions for her experience.

2. The formal structure of quantum mechanics is a theory of how the agent ought to organize her Bayesian probabilities for the consequences of all her potential actions on the things around her. Implicit in this is a theory of the structure of actions. The theory works as follows:

3. When the agent posits a system, she posits a Hilbert space $\mathcal{H}_d$ as the arena for all her considerations.

4. Actions upon the system are captured by positive-operator valued measures $\{E_i\}$ on $\mathcal{H}_d$. Potential consequences of the action are labeled by the individual elements E_i within the set(there is a formal similarity between this and the development in Cox (Cox, 1961), where "questions" are treated as sets, and "answers" are treated as elements within the sets), i.e.,

$$\text{action } = \{E_i\} \quad \text{and} \quad \text{consequence } = E_k$$

5. Quantum mechanics organizes the agent's beliefs by saying that she should strive to find a single density operator ρ such that her degrees

of belief will always satisfy

$$Prob(\text{consequence}|\text{action}) = Prob(E_k|\{E_i\}) = tr\,\rho E_k$$

no matter what action $\{E_i\}$is under consideration.

6. Unitary time evolution and more general quantum operations (completely positive maps) do not represent objective underlying dynamics, but rather address the agent's belief changes accompanying the flow of time, as well as belief changes consequent upon any actions taken.

7. When the agent posits *two* things external to herself, the arena for all her considerations becomes $\mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2}$. Actions and consequences now become POVMs on $\mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2}$.

8. The agent can nonetheless isolate the notion of an action on a single one of the things alone: These are POVMs of the from $\{E_i \otimes I\}$, and similarly with systems reversed $\{I \otimes E_i\}$.

9. Resolving the consequence of an action on *one* of the things may cause the agent to update her expectations for the consequences of any further actions she might take on the *other* thing. But for those latter consequences to come about, she must elicit them through an actual action on the second system.

The present discussion, of course, has predominantly focussed on Item 5 in this list, rewriting the point in purely probabilistic terms. With regard to the discussion in the present Section, however, the main points to note are Items 4, 7, 8, and 9. Regarding our usage of the word "measurement", they say that one should think of it simply as an action upon the system of interest. Actions lead to consequences within the experience of the agent, and that is what a quantum measurement is. A quantum measurement finds nothing, but very much makes something.

Thus, in a QBist painting of quantum mechanics, quantum measurements are "generative" in a very real sense. But by that turn, the consequences of our actions on physical systems must be egocentric as well. Measurement outcomes come about for the agent himself. Quantum mechanics is a single-user theory through and through - first in the usual Bayesian sense with regard to personal beliefs, and second in that quantum measurement outcomes are wholly personal experiences.

Of course, as a single-user theory, quantum mechanics is available to any agent to guide and better prepare him for his own encounters with the world. And although quantum mechanics has nothing to say about another agent's personal experiences, agents can communicate and use the information gained from each other to update their probability assignments. In the spirit of the Paulian Idea, however, querying another agent means taking an action on him. Whenever "I" encounter a quantum system, and take an action upon it, it catalyzes a con- sequence in my experience that my experience could not have foreseen. Similarly, by a Copernican-style principle, I should assume the same for "you": Whenever you encounter a quantum system, taking an action upon it, it catalyzes a consequence in your experience. By one category of thought, we are agents, but by another category of thought we are physical systems. And when we take actions upon each other, the category distinctions are symmetrical. Like with the Rubin vase, the best the eye can do is flit back and forth between the two formulations.

The previous paragraph should have made clear that viewing quantum mechanics as a single user theory does not mean there is only one user. QBism does not lead to solipsism. Any charge of solipsism is further refuted by two points central to the Paulian Idea. (Fuchs, 2002b). One is the conceptual split of the world into two parts - one an agent and the other an external quantum system - that gets the discussion of quantum measurement off the ground in the first place. If such a split were not needed for making sense of the question of actions (actions upon what? in what? with respect to what?), it would not have been made. Imagining a quantum measurement without an autonomous quantum system participating in the process would be as paradoxical as the Zen koan of the sound of a single hand clapping. The second point is that once the agent chooses an action $\{E_i\}$, the particular consequence E_k of it is beyond his control. That is to say, the particular outcome of a quantum measurement is not a product of his desires, whims, or fancies - this is the very reason he uses the calculus of probabilities in the first place: they quantify his uncertainty (Lindley, 2006), an uncertainty that, try as he might, he cannot get around. So, implicit in this whole picture - this whole Paulian Idea - is an "external world ... made of something", just as Martin Gardner calls for. It is only that quantum theory is a rather small theory: Its boundaries are set by being a handbook for agents immersed within that “world made of something”.

But a small theory can still have grand import, and quantum mechanics most certainly does. This is because it tells us how a user of the theory sees his role in the world. Even if quantum mechanics - viewed as an addition to probability theory - is not a theory of the world itself, it is certainly conditioned by the particular character of this world. Its empirical content is exemplified by the simplest case of the urgleichung,

$$q(j) = (d+1)\sum_{i=1}^{d^2} p(i)r(j|i) - 1$$

which takes this specific form rather than an infinity of other possibilities. Even though quantum theory is now understood as a theory of acts, decisions, and consequences (Savage, 1954), it tells us, in code, about the character of our particular world. Apparently, the world is made of a stuff that does not have “consequences” waiting around to fulfill our “actions” - it is a world in which the consequences are generated on the fly. One starts to get a sense of a world picture that is part personal - truly personal - and part the joint product of all that interacts. This is a world of indeterminism, but one with no place for “objective chance” in the sense of Lewis’ Principal Principle (Harper, 2012; Lewis, 1986a). From within any part, the future is undetermined. If one of those parts is an agent, then it is an agent in a situation of uncertainty. And where there is uncertainty, agents should use the calculus of Bayesian probability in order to make the best go at things.

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