

Gambler's Ruin Problem

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1 The Gambler's Ruin Problem

1.1 Pascal's problem of the gambler's ruin

Two gamblers, A and B , play with three dice. At each throw, if the total is 11 then B gives a counter to A ; if the total is 14 then A gives a counter to B . They start with twelve counters each, and the first to possess all 24 is the winner. What are their chances of winning?

Pascal gives the correct solution. The ratio of their respective chances of winning $p_A : p_B$, is

$$150094635296999122 : 129746337890625$$

which is the same as

$$282429536481 : 244140625$$

on dividing by 3^{12} .

Unfortunately it is not certain what method Pascal used to get this result.

1.2 Huygen's Approach

However, Huygens soon heard about this new problem, and solved it in a few days (sometime between September 28, 1656 and October 12, 1656). He used a version of Pascal's idea of value.

1.2.1 Digression on Probability and Expected Value

Many problems in chance are inextricably linked with numerical outcomes, especially in gambling and finance (where "numerical outcome" is a euphemism for money). In these cases probability is inextricably linked to "value", as we now show.

To aid our thinking let us consider an everyday concrete and practical problem

A plutocrat (member of a plutocracy = a government or state in which the wealthy rule) makes the following offer. She will flip a fair coin; if it shows

heads she will give you \$1, if it shows tails she will give your friend Jack \$1. What is the offer worth to you? That is to say, for what fair price $\$p$, should you sell it?

Clearly, whatever price $\$p$ this is worth to you, it is worth the same price $\$p$ to Jack, because the coin is fair, i.e., symmetrical (assuming he needs and values money just as much as you do). So, to the pair of you, this offer is altogether worth $\$2p$. But whatever the outcome, the plutocrat has given away \$1. Hence $\$2p = \1 , so that $p = \frac{1}{2}$ and the offer is worth $\frac{1}{2}$ to you.

It seems natural to regard this value $p = \frac{1}{2}$ as a measure of your chance of winning the money. It is intuitively reasonable to make the following general rule.

Suppose you receive \$1 with probability p (and otherwise you receive nothing). Then the *value* or *fair price* of this offer is $\$p$. More generally, if you receive $\$d$ with probability p (and nothing otherwise) then the *fair price* or *expected value* of this offer is given by

$$\text{expected value} = pd$$

This simple idea turns out to be enormously important later on; for the moment we note only that it is certainly consistent with ideas about probability. For example, if the plutocrat definitely gives you \$1 then this is worth exactly \$1 to you, and $p = 1$. Likewise, if you are definitely given nothing, then $p = 0$ and it is easy to see that $0 \leq p \leq 1$, for any such offers.

In particular, for the specific example above we find that the probability of a head when a fair coin is flipped is $\frac{1}{2}$. Likewise a similar argument shows that the probability of a six when a fair die is rolled is $\frac{1}{6}$ (simply imagine the plutocrat giving \$1 to one of six people selected by the roll of the die).

The “fair price” of such offers is often called the *expected value*, or *expectation*, to emphasize its chance nature.

Returning to Huygens- Huygen’s definition of value

if you are offered $\$x$ with probability p , or $\$y$ with probability q ($p + q = 1$), then the value of this offer to you is $(px + qy)$.

Now of course we do not know for sure if this was Pascal's method, but Pascal was certainly at least as capable of extending his own ideas as Huygens was. The correct balance of probabilities indicates that he did use this method. By long-standing tradition this problem is always solved in books on elementary probability, and so we now give a modern version of the solution. Here is a general statement of the problem.

1.3 Problem and Solution

Two players, A and B again, play a series of independent games. Each game is won by A with probability α , or by B with probability β ; the winner of each game gets one counter from the loser. Initially A has m counters and B has n . The victor of the contest is the first to have all $m + n$ counters, the loser is said to be "ruined", which explains the name of the problem. What are the chances of A and B to be the victor?

Note that $\alpha + \beta = 1$, and for the moment we assume that $\alpha \neq \beta$.

We will need the "partition rule" for two events from probability theory:

$$P(A) = P(A|B)P(B) + P(A| \neg B)P(\neg B)$$

and the "conditional partition rule" for three events

$$P(A|C) = P(A|B \cap C)P(B|C) + P(A| \neg B \cap C)P(\neg B|C)$$

It is also helpful to discuss a different problem first.

1.3.1 Digression to the Problem of Points

"Prolonged gambling differentiates people into two groups, those playing with the odds, who are following a trade or profession; and those playing against the odds, who are indulging a hobby or pastime, and if this involves a regular annual outlay, this is no more than what has been said about most other amusements."

John Venn

We consider here a problem which was of particular importance in the history and development of probability. In most texts on probability, one starts off

with problems involving dice, cards, and other simple gambling devices. The application of the theory is so natural and useful that it might be supposed that the creation of probability paralleled the creation of dice and cards. In fact, this is far from being the case. The greets single initial step in constructing a theory of probability was made in response to a more obscure question - *the problem of points*.

Roughly speaking the essential question is this.

Two players, traditionally called A and B , are competing for a prize. The contest takes the form of a sequence of independent similar trials; as a result of each trial one of the contestants is awarded a point. The first player to accumulate n points is the winner; in colloquial parlance A and B are playing the best of $2n - 1$ points. Tennis matches are usually the best of five sets; $n = 3$.

The problem arises when the contest has to be stopped or abandoned before either has won n points; in fact A needs a points (having $n - a$ already) and B needs b points (having $n - b$ already). How should the prize be fairly divided? (Typically the “prize” consisted of stakes put up by A and B , and held by the stakeholder).

For example, in tennis, sets correspond to points and men play the best of five sets. If the players were just beginning the fourth set when the court was swallowed up by an earthquake, say, what would be a fair division of the prize? (assuming a natural reluctance to continue the game on some nearby court).

This is a problem of great antiquity; it first appeared in print in 1494 in a book by Luca Pacioli, but was almost certainly an old problem even then. In his example, A and B were playing the best of 11 games for a prize of ten ducats, and are forced to abandon the game when A has 5 points (needing 1 more) and B has 2 points (needing 4 more). How should the prize be divided?

Though Pacioli was a man of great talent (among many other things his book included the first printed account of double-entry book-keeping), he could not solve this problem.

Nor could Tartaglia (who is best known for showing how to find the roots of a cubic equation), nor could Forestani, Peverone, or Cardano, who all made attempts during the 16th century.

In fact, the problem was finally solved by Blaise Pascal in 1654, who, with Fermat thereby officially inaugurated the theory of probability. In that year, probably sometime around Pascal's birthday (June 19; he was 31), the problem of points was brought to his attention. The enquiry was made by Chevalier de Méré (Antoine Gombaud) who as a man-about-town and a gambler, had a strong and direct interest in the answer. Within a very short time, Pascal had solved the problem in two different ways. In the course of a correspondence with Fermat, a third method of solution was found by Fermat.

Two of these methods relied on counting the number of equally likely outcomes. Pascal's great step forward was to create a method that did not rely on having equally likely outcomes. This breakthrough came about as a result of his explicit formulation of the idea of the value of a bet or lottery, as we discussed above, i.e., if you have a probability p of winning \$1 then the game is worth $\$p$ to you.

It naturally follows that, in the problem of the points, the prize should be divided in proportion to the player's respective probabilities of winning if the game were to be continued. The problem is therefore more precisely stated thus.

Precise problem of the points. A sequence of fair coins is flipped; A get a pony for every head, B a point for every tail. Player A wins if there are a heads before b tails, otherwise B winds. Find the probability that A wins.

Solution. Let $\alpha(a, b)$ be the probability that A wins and $\beta(a, b)$ the probability that B wins. If the first flip is a head, then A now needs $a - 1$ further heads to win, so the conditional probability that A wins, given a head, is $\alpha(a - 1, b)$. Likewise the conditional probability that A wins, given a tail is $\alpha(a, b - 1)$. Hence, by the partition rule,

$$\alpha(a, b) = \frac{1}{2}\alpha(a - 1, b) + \frac{1}{2}\alpha(a, b - 1) \quad (1)$$

Thus, if we know $\alpha(a, b)$ for small values of a and b , we can find the solution for any a and b by simple recursion. And of course we do know such values

of $\alpha(a, b)$, because if $a = 0$ and $b > 0$, then A has won and takes the whole prize: that is to say

$$\alpha(0, b) = 1 \quad (2)$$

Likewise, if $b = 0$ and $a > 0$, then B has won, and so

$$\alpha(a, 0) = 0 \quad (3)$$

How do we solve (1) in general, with (2) and (3)? Recall the fundamental property of Pascal's triangle: the entries in the (row, column):

$$\begin{aligned} (0, 0) \binom{0}{0} &= 1 \\ (1, 0) \binom{1}{0} &= 1 \quad (1, 1) \binom{1}{1} = 1 \\ (2, 0) \binom{2}{0} &= 1 \quad (2, 1) \binom{2}{1} = 2 \quad (2, 2) \binom{2}{2} = 1 \end{aligned}$$

and so on, i.e., the entries $c(j + k, k) = d(j, k)$ satisfy

$$d(j, k) = d(j - 1, k) + d(j, k - 1) \quad (4)$$

You do not need to be genius to suspect that the solution $\alpha(a, b)$ of (1) is going to be connected with the solutions

$$c(j + k, k) = d(j, k) = \binom{j + k}{k}$$

of (4). We can make the connect even more transparent by writing

$$\alpha(a, b) = \frac{1}{2^{a+b}} u(a, b)$$

Then (1) becomes

$$u(a, b) = u(a - 1, b) + u(a, b - 1) \quad (5)$$

with

$$u(0, b) = 2^b \quad \text{and} \quad u(a, 0) = 0 \quad (6)$$

There are various ways of solving (5) with conditions (6), but Pascal had the inestimable advantage of having already obtained the solution by another method. Thus, he had simply to check that the answer is indeed

$$u(a, b) = 2 \sum_{k=0}^{b-1} \binom{a+b-1}{k} \quad (7)$$

and

$$\alpha(a, b) = \frac{1}{2^{a+b-1}} \sum_{k=0}^{b-1} \binom{a+b-1}{k} \quad (8)$$

At long last there was a solution to this classic problem. We may reasonably ask why Pascal was able to solve it in a matter of weeks, when all previous attempts had failed for at least 150 years. As usual the answer lies in a combination of circumstances: mathematicians had become better at counting things; the binomial coefficients were better understood; notation and the techniques of algebra had improved immeasurably; and Pascal had a couple of very good ideas.

Pascal immediately realized the power of these ideas and techniques and quickly invented new problems on which to use them.

The Gambler's Ruin problem is the best known of them.

Just as in the problem of points, suppose that at some stage A has a counters (so B has $m+n-a$ counters), and let A 's chances of victory at that point be $v(a)$. If A wins the next game his chance of victory is now $v(a+1)$; if A loses the next game his chance of victory is $v(a-1)$. Hence by the partition rule,

$$v(a) = \alpha v(a+1) + \beta v(a-1) \quad , \quad 1 \leq a \leq m+n-1 \quad (9)$$

Furthermore we know that

$$v(m+n) = 1 \quad (10)$$

because in this case A has all the counters, and

$$v(0) = 0 \quad (11)$$

because A then has no counters.

1.3.2 Solving difference equations

Any sequence $(x_r; r \geq 0)$ in which each term is a function of its predecessors, so that

$$x_{r+k} = f(x_r, x_{r+1}, \dots, x_{r+k-1}) \quad , \quad r \geq 0 \quad (12)$$

is said to satisfy the *recurrence relation* (12). When f is linear this is called a difference equation of order k :

$$x_{r+k} = a_0 x_r + a_1 x_{r+1} + \dots + a_{k-1} x_{r+k-1} + g(r) \quad , \quad a_0 \neq 0 \quad (13)$$

When $g(r) = 0$, the equation is homogeneous:

$$x_{r+k} = a_0 x_r + a_1 x_{r+1} + \dots + a_{k-1} x_{r+k-1} \quad , \quad a_0 \neq 0 \quad (14)$$

Some examples to give a hint at the solutions are:

If we have

$$p_n = qp_{n-1}$$

the solution is easy because

$$p_{n-1} = qp_{n-2} \quad , \quad p_{n-2} = qp_{n-3}$$

and so on. By successive substitution we obtain

$$p_n = q^n p_0$$

If we have

$$p_n = (q - p)p_{n-1} + p$$

this is almost as easy to solve when we notice that

$$p_n = \frac{1}{2}$$

is a particular solution. Now writing

$$p_n = \frac{1}{2} + x_n$$

gives

$$x_n = (q - p)x_{n-1} = (q - p)^n x_0$$

so that

$$p_n = \frac{1}{2} + (q - p)^n x_0$$

If we have

$$p_n = pp_{n-1} + pqp_{n-2} \quad , \quad pq \neq 0$$

is more difficult, but after some work which we omit, the solution turns out to be

$$p_n = c_1 \lambda_1^n + c_2 \lambda_2^n \tag{15}$$

where λ_1 and λ_2 are the roots of

$$x^2 + px - pq = 0$$

and c_1 and c_2 are arbitrary constants.

Having seen these special examples, one is not surprised to see the general solution to the second order difference equation: let

$$x_{r+2} = a_0 x_r + a_1 x_{r+1} + g(r) \quad , \quad r \geq 0 \tag{16}$$

Suppose that $\pi(r)$ is any function such that

$$\pi(r+2) = a_0 \pi(r) + a_1 \pi(r+1) + g(r)$$

and suppose that λ_1 and λ_2 are the roots of

$$x^2 = a_0 + a_1 x$$

Then the solution of (16) is given by

$$x_r = \begin{cases} c_1 \lambda_1^r + c_2 \lambda_2^r + \pi(r) & \lambda_1 \neq \lambda_2 \\ (c_1 + c_2 r) \lambda_1^r + \pi(r) & \lambda_1 = \lambda_2 \end{cases}$$

where c_1 and c_2 are arbitrary constants. Here $\pi(r)$ is called a *particular solution*. One should note that λ_1 and λ_2 may be complex, as then may c_1 and c_2 .

The solution of higher-order difference equations proceeds along similar lines. From section 1.3.2 we know that the solution of (9) takes the form

$$v(a) = c_1 \lambda^a + c_2 \mu^a$$

where c_1 and c_2 are arbitrary constants, and λ and μ are roots of

$$\alpha x^2 - x + \beta = 0 \tag{17}$$

Trivially, the roots of (17) are $\lambda = 1$ and $\mu = \beta/\alpha \neq 1$ (since we assumed that $\alpha \neq \beta$). Hence, using (10) and (11), we find that

$$v(a) = \frac{1 - (\beta/\alpha)^a}{1 - (\beta/\alpha)^{m+n}} \quad (18)$$

In particular, when A starts with m counters,

$$p_A = v(m) = \frac{1 - (\beta/\alpha)^m}{1 - (\beta/\alpha)^{m+n}} \quad (19)$$

This method of solution of difference equation was unknown in 1656, so other approaches were employed. In obtaining the answer to the gambler's ruin problem, Huygens (and later workers) used intuitive induction with the proof omitted. Pascal probably did use (9) but solved it by a different route.

Finally, we consider the case when $\alpha = \beta$. Now (9) is

$$v(a) = \frac{1}{2}v(a+1) + \frac{1}{2}v(a-1) \quad (20)$$

and it is easy to check that, for arbitrary constants c_1 and c_2

$$v(a) = c_1 + c_2 a$$

satisfies (20). Now using (10) and (11) gives

$$v(a) = \frac{a}{m+n} \quad (21)$$

Thus, there is always a winner (hence the ruin of the loser) . Note that this relation is linear in a !

2 Ruin, Random Walks and Diffusion

2.1 General Orientation

Some of this discussion will be a repeat of result in section 1. Consider the familiar gambler who wins or loses a dollar with probability p and q , respectively. Let his initial capital be z and let him play against an adversary with initial capital $a - z$, so that the combined capital is a . The game continues until the gambler's capital either is reduced to zero or has increased to a , that

is, until one of the two players is ruined. We are interested in the probability of the gambler's ruin and the probability distribution of the duration of the game. This is *the classical ruin problem*.

Physical applications and analogies suggest the more flexible interpretation in terms of the notion of a variable point or “*particle*” on the x -axis. This particle starts from the *initial position* z , and moves at regular time intervals a unit step in the positive or negative direction, depending on whether the corresponding trial resulted in success or failure. The position of the particle after n steps represents the gambler's capital at the conclusion of the n^{th} trial. The trials terminate when the particle for the first time reaches either 0 or a , and we describe this by saying that the particle performs *a random walk with absorbing barriers at 0 and a* . This random walk is *restricted* to the possible positions $1, 2, \dots, a - 1$; in the absence of absorbing barriers the random walk is called *unrestricted*. Physicists use the random-walk model as a crude approximation to one-dimensional diffusion or Brownian motion, where a physical particle is exposed to a great number of molecular collisions which impart to it a random motion. The case $p > q$ corresponds to a *drift* to the right when shocks from the left are more probable; when $p = q = \frac{1}{2}$; the random walk is called *symmetric*.

In the limiting case $a \rightarrow \infty$ we get a random walk on a semi-infinite line: A particle starting at $z > 0$ performs a random walk up to the moment when it for the first time reaches the origin. In this formulation we recognize the *first-passage time problem*; it can be solved by elementary methods (at least for the symmetric case) and by the use of generating functions. The present derivation is self-contained.

In this section we shall use the method of *difference equations* which serves as an introduction to the differential equations of diffusion theory. This analogy leads in a natural way to various modifications and generalizations of the classical ruin problem, a typical and instructive example being the replacing of absorbing barriers by *reflecting* and *elastic* barriers. To describe a reflecting barrier, consider a random walk in a finite interval as defined before except that whenever the particle is at point 1 it has probability p of moving to position 2 and probability q to stay at 1. In gambling terminology this corresponds to a convention that whenever the gambler loses his last dollar it is generously replaced by his adversary so that the game can continue.

The physicist imagines a wall placed at the point $\frac{1}{2}$ of the as -axis with the property that a particle moving from 1 toward 0 is reflected at the wall and returns to 1 instead of reaching 0. Both the absorbing and the reflecting barriers are special cases of the so-called elastic barrier. We define an *elastic barrier at the origin* by the rule that from position 1 the particle moves with probability p to position 2; with probability δq it stays at 1; and with probability $(1 - \delta)q$ it moves to 0 and is absorbed (i.e., the process terminates). For $\delta = 0$ we have the classical ruin problem or absorbing barriers, for $\delta = 1$ reflecting barriers. As δ runs from 0 to 1 we have a family of intermediate cases. The greater δ is, the more likely is the process to continue, and with two reflecting barriers the process can never terminate.

Sections 2.2 and 2.3 are devoted to an elementary discussion of the classical ruin problem and its implications. The next three sections are more technical (and may be omitted); in 2.4 and 2.5 we derive the relevant generating functions and from them explicit expressions for the distribution of the duration of the game, etc. Section 2.6 contains an outline of the passage to the limit to the diffusion equation (the formal solutions of the latter being the limiting distributions for the random walk).

In section 2.7 the discussion again turns elementary and is devoted to *random walks in two or more dimensions* where new phenomena are encountered. Section 2.8 treats a generalization of an entirely different type, namely a random walk in one dimension where the particle is no longer restricted to move in unit steps but is permitted to change its position in jumps which are arbitrary multiples of unity.

2.2 The Classical Ruin problem

We shall consider the problem stated at the opening of the present section. Let q_z , be the probability of the gambler's ultimate ruin (Strictly speaking, the probability of ruin is defined in a sample space of infinitely prolonged games, but we can work with the sample space of n trials. The probability of ruin in less than n trials increases with n and has therefore a limit. We call this *limit* "the probability of ruin". All probabilities in this section may be interpreted in this way without reference to infinite spaces ruin and p_z the probability of his winning. In random-walk terminology q_z and p_z are the probabilities that a particle starting at z will be absorbed at 0 and a , respectively. We

shall show that $p_z + q_z = 1$, so that we need not consider the possibility of an unending game.

After the first trial the gambler's fortune is either $z - 1$ or $z + 1$, and therefore we must have

$$q_z = pq_{z+1} + qq_{z-1} \quad (22)$$

provided that $1 < z < a - 1$. For $z = 1$ the first trial may lead to ruin, and (22) is to be replaced by $q_1 = pq_2 + q$. Similarly, for $z = a - 1$ the first trial may result in victory, and therefore $q_{a-1} = qq_{a-2}$. To unify our equations we define

$$q_0 = 1 \quad , \quad q_a = 0 \quad (23)$$

With this convention the probability q_z of ruin satisfies (22) for $z = 1, 2, \dots, a - 1$.

Systems of the form (22) are known as *difference equations*, and (23) represents the *boundary conditions* on q_z . We shall derive an explicit expression for q_z by the *method of particular solutions*, which will also be used in more general cases.

Suppose first that $p \neq q$. It is easily verified that the difference equations (22) admit of the two particular solutions $q_z = 1$ and $q_z = (q/p)^z$. It follows that for arbitrary constants A and B the sequence

$$q_z = A + B \left(\frac{q}{p} \right)^z \quad (24)$$

represents a formal solution of (22). The boundary conditions (23) will hold if, and only if, A and B satisfy the two linear equations $A + B = 1$ and $A + b(q/p)^z = 0$. Thus

$$q_z = \frac{(q/p)^a - (q/p)^z}{(q/p)^a - 1} \quad (25)$$

is a formal solution of the difference equation (22), satisfying the boundary conditions (23). In order to prove that (24) is the required probability of ruin it remains to show that the solution is unique, that is, that *all* solutions of (22) are of the form (24). Now, given an arbitrary solution of (22), the two constants A and B can be chosen so that (24) will agree with it for $z = 0$ and $z = 1$. From these two values all other values can be found by substituting in (22) successively $z = 1, 2, 3, \dots$. Therefore two solutions which agree for

$z = 0$ and $z = 1$ are identical, and hence every solution is of the form (24).

Our argument breaks down if $p = q = \frac{1}{2}$, for then (25) is meaningless because in this case the two formal particular solutions $q_z = 1$ and $q_z = (q/p)^z$ are identical. However, when $p = q = \frac{1}{2}$ we have a formal solution in $q_z = z$, and therefore $q_z = A + Bz$ is a solution of (22) depending on two constants. In order to satisfy the boundary conditions (23) we must put $A = 1$ and $A + Ba = 0$. Hence

$$q_z = 1 - \frac{z}{a} \quad (26)$$

(The same numerical value can be obtained formally from (25) by finding the limit as $p \rightarrow \frac{1}{2}$, using L'Hospital's rule).

We have thus proved that the required *probability of the gambler's ruin* is given by (25) if $p \neq q$ and by (26) if $p = q = \frac{1}{2}$. The probability p_z of the gambler's winning the game equals the probability of his adversary's ruin and is therefore obtained from our formulas on replacing p , q , and z by q , p , and $a - z$, respectively. It is readily seen that $p_z + q_z = 1$, as stated previously.

We can reformulate our result as follows: *Let a gambler with an initial capital z play against an infinitely rich adversary who is always willing to play, although the gambler has the privilege of stopping at his pleasure. The gambler adopts the strategy of playing until he either loses his capital or increases it to a (with a net gain $a - z$). Then q_z is the probability of his losing and $1 - q_z$ the probability of his winning.*

Under this system the gamblers ultimate gain or loss is a random variable \mathbf{G} which assumes the values $a - z$ and $-z$ with probabilities $1 - q_z$, and q_z , respectively. The expected gain is

$$\mathbf{E}(\mathbf{G}) = a(1 - q_z) - z \quad (27)$$

Clearly $\mathbf{E}(\mathbf{G}) = 0$ if, and only if, $p = q$. This means that, with the system described, a *"fair" game remains fair, and no "unfair" game can be changed into a "fair" one.*

From (26) we see that in the case $p = q$ a player with initial capital $z = 999$ has a probability 0.999 to win a dollar before losing his capital. With $q = 0.6$, $p = 0.4$ the game is unfavorable indeed, but still the probability (25) of winning a dollar before losing the capital is about $\frac{2}{3}$. In general, a gambler with a

relatively large initial capital z has a reasonable chance to win a small amount $a - z$ before being ruined (A certain man used to visit Monte Carlo year after year and was always successful in recovering the cost of his vacations. He firmly believed in a magic power over chance. Actually his experience is not surprising. Assuming that he started with ten times the ultimate gain, the chances of success in any year are nearly 0.9. The probability of an unbroken sequence of ten successes is about $(1 - \frac{1}{2})^{10} \approx e^{-1} \approx 0.37$. Thus continued success is by no means improbable. Moreover, one failure would, of course, be blamed on an oversight or momentary indisposition).

Let us now investigate the effect of changing stakes. Changing the unit from a dollar to a half dollar is equivalent to doubling the initial capitals. The corresponding probability of ruin q_z^* is obtained from (25) on replacing z by $2z$ and a by $2a$:

$$q_z = \frac{(q/p)^{2a} - (q/p)^{2z}}{(q/p)^{2a} - 1} = q_z \cdot \frac{(q/p)^{2a} + (q/p)^{2z}}{(q/p)^{2a} + 1} \quad (28)$$

For $q > p$ the last fraction is greater than unity and $q_z^* > q_z$. We restate this conclusion as follows: *if the stakes are doubled while the initial capitals remain unchanged, the probability of ruin decreases for the player whose probability of success is $p < \frac{1}{2}$ and increases for the adversary (for whom the game is advantageous)*. Suppose, for example, that Peter owns 90 dollars and Paul 10, and let $p = 0.45$, the game being unfavorable to Peter. If at each trial the stake is one dollar, table 1 shows the probability of Peter's ruin to be 0.866, approximately. If the same game is played for a stake of 10 dollars, the probability of Peter's ruin drops to less than one fourth, namely about 0.210. Thus the effect of increasing stakes is more pronounced than might be expected. In general, if k dollars are staked at each trial, we find the probability of ruin from (25), replacing z by Z/k and a by a/k ; the probability of ruin decreases as k increases. In a game with constant stakes the unfavored gambler minimizes the probability of ruin by selecting the stake as large as consistent with his goal of gaining an amount fixed in advance. The empirical validity of this conclusion has been challenged, usually by people who contended that every "unfair" bet is unreasonable. If this were to be taken seriously, it would mean the end of all insurance business, for the careful driver who insures against liability obviously plays a game that is technically "unfair". Actually, there exists no theorem in probability to discourage such a driver from taking insurance.

p	q	z	a	$Prob(Ruin)$	$Prob(Success)$	$Expected\ Gain$	$Expected\ Duration$
0.5	0.5	9	10	0.1	0.9	0	9
0.5	0.5	90	100	0.1	0.9	0	900
0.5	0.5	900	1000	0.1	0.9	0	90000
0.5	0.5	950	1000	0.05	0.95	0	47500
0.5	0.5	8000	10000	0.2	0.8	0	16000000
0.45	0.55	9	10	0.210	0.790	-1.1	11
0.45	0.55	90	100	0.866	0.134	-76.6	765.6
0.45	0.55	99	100	0.182	0.818	-17.2	171.8
0.4	0.6	90	100	0.983	0.017	-88.3	441.3
0.4	0.6	99	100	0.333	0.667	-32.3	161.7

Table 1: Illustrating the Classical Ruin Problem. The initial capital is z . The game terminates with ruin (loss z) or capital a (gain $a - z$).

2.2.1 The limiting case

The limiting case $a = \infty$ corresponds to a game against an infinitely rich adversary. Letting $a \rightarrow \infty$ in (25) and (26) we get

$$q_z = \begin{cases} 1 & \text{if } p \leq q \\ (q/p)^z & \text{if } p > q \end{cases} \quad (29)$$

We interpret q_z as the probability of ultimate ruin of a gambler with initial capital z playing against an infinitely rich adversary (It is easily seen that the q_z represent a solution of the difference equations (22) satisfying the (now unique) boundary condition $q_0 = 1$. When $p > q$ the solution is not unique). In random walk terminology q_z is the probability that a particle starting at $z > 0$ will ever reach the origin. It is more natural to rephrase this result as follows; *In a random walk starting at the origin the probability of ever reaching the position $z > 0$ equals 1 if $p \geq q$ and equals $(p/q)^z$ when $p < q$.*

2.3 Expected Duration of the Game

The probability distribution of the duration of the game will be deduced in the following sections. However, its expected value can be derived by a much simpler method which is of such wide applicability that it will now be explained at the cost of a slight duplication.

We are still concerned with the classical ruin problem formulated at the beginning of this section. We shall assume as known the fact that the duration

of the game has a finite expectation D_z . A rigorous proof will be given in the next section. If the first trial results in success the game continues as if the initial position had been $z + 1$. The conditional expectation of the duration assuming success at the first trial is therefore $D_{z+1} + 1$. This argument shows that the expected duration D_z satisfies the difference equation

$$D_z = pD_{z+1} + qD_{z-1} + 1 \quad , \quad 0 < z < a \quad (30)$$

with the boundary conditions

$$D_0 = 0 \quad , \quad D_a = 0 \quad (31)$$

The appearance of the term 1 makes the difference equation (30) non-homogeneous. If $p \neq q$, then $D_z = z/(q - p)$ is a formal solution of (30). The difference Δ_z of any two solutions of (30) satisfies the homogeneous equations $\Delta_z = p\Delta_{z+1} + q\Delta_{z-1}$, and we know already that all solutions of this equation are of the form $A + B(q/p)^z$. It follows that when $p \neq q$ all solutions of (30) are of the form

$$D_z = \frac{z}{q - p} + A + B \left(\frac{q}{p} \right)^z \quad (32)$$

The boundary conditions (31) require that

$$A + B = 0 \quad , \quad A + B \left(\frac{q}{p} \right)^a = -\frac{a}{q - p}$$

Solving for A and B , we find

$$D_z = \frac{z}{q - p} - \frac{a}{q - p} \cdot \frac{1 - (q/p)^z}{1 - (q/p)^a} \quad (33)$$

Again the method breaks down if $q = p = \frac{1}{2}$. In this case we replace $z/(q - p)$ by $-z^2$, which is now a solution of (30). It follows that when $q = p = \frac{1}{2}$ all solutions of (30) are of the form $D_z = -z^2 + A + Bz$. The required solution D_z satisfying the boundary conditions (31) is

$$D_z = z(a - z) \quad (34)$$

The expected duration of the game in the classical ruin problem is given by (33) or (34), according as $p \neq q$ or $q = p = \frac{1}{2}$.

It should be noted that this duration is considerably longer than we would

naively expect. If two players with 500 dollars each toss a coin until one is ruined, the average duration of the game is 250,000 trials. If a gambler has only one dollar and his adversary 1000, the average duration is 1000 trials. Further examples are found in table 1.

As indicated at the end of the preceding section, we may pass to the limit $a \rightarrow \infty$ and consider a game against an infinitely rich adversary. When $p > q$ the game may go on forever, and in this case it makes no sense to talk about its expected duration. When $p < q$ we get for the expected duration $z(q - p)^{-1}$, but when $p = q$ the expected duration is infinite. (The same result was established will be proved independently in the next section.)

2.4 Generating Functions for the Duration of the Game and for the First-Passage Times

We shall use the method of generating functions to study the duration of the game in the classical ruin problem, that is, the restricted random walk with absorbing barriers at 0 and a . The initial position is z (with $0 < z < a$). Let $u_{z,n}$ denote the probability that the process ends with the n^{th} step at the barrier 0 (gamblers ruin at the n^{th} trial). After the first step the position is $z + 1$ or $z - 1$, and we conclude that for $1 < z < a - 1$ and $n \geq 1$

$$u_{z,n+1} = pu_{z+1,n} + qu_{z-1,n} \quad (35)$$

This is a difference equation analogous to (22), but depending on the two variables z and n . In analogy with the procedure of section 2.2 we wish to define boundary values $u_{0,n}$, $u_{a,n}$, and $u_{z,0}$ so that (35) becomes valid also for $z = 1$, $z = a - 1$, and $n = 0$. For this purpose we put

$$u_{0,n} = u_{a,n} = 0 \quad \text{when } n \geq 1 \quad (36)$$

and

$$u_{0,0} = 1, \quad u_{z,0} = 0 \quad \text{when } 0 < z \leq a \quad (37)$$

Then (35) holds for all z with $0 < z < a$ and all $n \geq 0$.

We now introduce the generating functions

$$U_z(s) = \sum_{n=0}^{\infty} u_{z,n} s^n \quad (38)$$

Multiplying (35) by s^{n+1} and adding for $n = 0, 1, 2, \dots$, we find

$$U_z(s) = psU_{z+1}(s) + qsU_{z-1}(s) \quad 0 < z < a \quad (39)$$

the boundary conditions (36) and (37) lead to

$$U_0(s) = 1 \quad , \quad U_a(s) = 0 \quad (40)$$

The system (39) represents difference equations analogous to (22), and the boundary conditions (40) correspond to (23). The novelty lies in the circumstance that the coefficients and the unknown $U_z(s)$ now depend on the variable s , but as far as the difference equation is concerned, s is merely an arbitrary constant. We can again apply the method of section 2.2 provided we succeed in finding two particular solutions of (39). It is natural to inquire whether there exist two solutions $U_z(s)$ of the form $U_z(s) = \lambda^z(s)$. Substituting this expression into (39), we find that $\lambda(s)$ must satisfy the quadratic equation

$$\lambda(s) = ps\lambda^2(s) + qs \quad (41)$$

which has the two roots

$$\lambda_1(s) = \frac{1 + \sqrt{1 - 4pqs^2}}{2ps} \quad , \quad \lambda_2(s) = \frac{1 - \sqrt{1 - 4pqs^2}}{2ps} \quad (42)$$

(we take $0 < s < 1$ and the positive square root).

We are now in possession of two particular solutions of (39) and conclude as in section 2.2 that every solution is of the form

$$U_z(s) = A(s)\lambda_1^z(s) + B(s)\lambda_2^z(s) \quad (43)$$

with $A(s)$ and $B(s)$ arbitrary. To satisfy the boundary conditions (40), we must have $A(s) + B(s) = 1$ and $A(s)\lambda_1^a(s) + B(s)\lambda_2^a(s) = 0$, whence

$$U_z(s) = \frac{\lambda_1^a(s)\lambda_2^z(s) - \lambda_1^z(s)\lambda_2^a(s)}{\lambda_1^a(s) - \lambda_2^a(s)} \quad (44)$$

Using the obvious relation $\lambda_1(s)\lambda_2(s) = q/p$, this simplifies to

$$U_z(s) = \left(\frac{q}{p}\right)^z \frac{\lambda_1^{a-z}(s) - \lambda_2^a(s)}{\lambda_1^{a-z}(s) - \lambda_2^a(s)} \quad (45)$$

This is the required generating function of the probability of ruin (absorption at 0) at the n^{th} trial. The same method shows that the generating function for the probabilities of absorption at a is given by

$$U_z(s) = \frac{\lambda_1^z(s) - \lambda_2^z(s)}{\lambda_1^a(s) - \lambda_2^a(s)} \quad (46)$$

The generating function for the duration of the game is, of course, the sum of the generating functions (45) and (46).

2.4.1 Infinite Intervals and First Passages

The preceding considerations apply equally to random walks on the interval $(0, \infty)$ with an absorbing barrier at 0. A particle starting from the position $z > 0$ is eventually absorbed at the origin or else the random walk continues forever. Absorption corresponds to the ruin of a gambler with initial capital z playing against an infinitely rich adversary. The generating function $U_z(s)$ of the probabilities $u_{z,n}$ that absorption takes place exactly at the n^{th} trial satisfies again the difference equations (39) and is therefore of the form (43), but this solution is unbounded at infinity unless $A(s) = 0$. The other boundary condition is now $U_0(s) = 1$, and hence $B(s) = 1$ or

$$U_z(s) = \lambda_2^z(s) \quad (47)$$

[The same result can be obtained by letting $a \rightarrow \infty$ in (45), and remembering that $\lambda_1(s)\lambda_2(s) = q/p$.]

It follows from (47) for $s = 1$ that an ultimate absorption is certain if $p \leq q$, and has probability $(q/p)^z$ otherwise. The same conclusion was reached in section 2.2.

Our absorption at the origin admits of an important alternative interpretation as a first passage in an *unrestricted* random walk. Indeed, on moving the origin to the position z it is seen that in a random walk on the entire line and starting from the origin $u_{z,n}$ is the probability that the first visit to the point $-z < 0$ takes place at the n^{th} trial. That the corresponding generating function (47) is the z^{th} power of λ_2 reflects the obvious fact that the waiting time for the first passage through $-z$ is the sum of z independent waiting times between the successive first passages through $-1, -2, \dots, -z$.

2.5 Explicit Expressions

The generating function U_z , of (45) depends formally on a square root but is actually a rational function. In fact, an application of the binomial theorem reduces the denominator to the form

$$\lambda_1^a(s) - \lambda_2^a(s) = s^{-a} \sqrt{1 - 4pqs^2} P_a(s) \quad (48)$$

where P_a is an even polynomial of degree $a - 1$ when a is odd, and of degree $a - 2$ when a is even. The numerator is of the same form except that a is replaced by $a - z$. Thus U_z is the ratio of two polynomials whose degrees differ at most by 1. Consequently it is possible to derive an explicit expression for the ruin probabilities $u_{z,n}$ by the method of partial fractions. The result is interesting because of its connection with diffusion theory, and the derivation as such provides an excellent illustration for the techniques involved in the practical use of partial fractions.

The calculations simplify greatly by the use of an auxiliary variable ϕ defined by

$$\cos \phi = \frac{1}{2\sqrt{pq} \cdot s} \quad (49)$$

(To $0 < s < 1$ there correspond complex values of ϕ , but this has no effect on the formal calculations.) From (42)

$$\lambda_1(s) = \sqrt{q/p} [\cos \phi + i \sin \phi] = \sqrt{q/p} e^{i\phi} \quad (50)$$

while $\lambda_2(s)$ equals the right side with i replaced by $-i$. Accordingly,

$$U_z(s) = (\sqrt{q/p})^z \frac{\sin(a-z)\phi}{\sin a\phi} \quad (51)$$

The roots s_1, s_2, \dots of the denominator are simple and hence there exists a partial fraction expansion of the form

$$(\sqrt{q/p})^z \frac{\sin(a-z)\phi}{\sin a\phi} = A + Bs + \frac{\rho_1}{s_1 - s} + \dots + \frac{\rho_{a-1}}{s_{a-1} - s} \quad (52)$$

In principle we should consider only the roots s_v which are not roots of the numerator also, but if s_v is such a root then $U_z(s)$ is continuous at $s = s_v$ and hence $\rho_v = 0$. Such canceling roots therefore do not contribute to the

right side and hence it is not necessary to treat them separately.

The roots s_1, s_2, \dots, s_{a-1} correspond obviously to $\phi_v = \pi v/a$ with $v = 1, \dots, a-1$, and so

$$s_v = \frac{1}{2\sqrt{pq} \cos \pi v/a} \quad (53)$$

This expression makes no sense when $v = a/2$ and a is even, but then ϕ_v is a root of the numerator also and this root should be discarded. The corresponding term in the final result vanishes, as is proper.

To calculate ρ_v we multiply both sides in (52) by $s_v - s$ and let $s \rightarrow s_v$. Remembering that $\sin a\phi_v = 0$ and $\cos a\phi_v = 1$ we get

$$\rho_v = (\sqrt{q/p})^z \sin z\phi_v \cdot \lim_{s \rightarrow s_v} \frac{s - s_v}{\sin a\phi}$$

The last limit is determined by L'Hospital's rule using implicit differentiation in (50). The result is

$$\rho_v = a^{-1} \cdot 2\sqrt{pq} (\sqrt{q/p})^z \sin z\phi_v \cdot \sin \phi_v \cdot s_v^2$$

From the expansion of the right side in (53) into geometric series we get for $n > 1$

$$u_{z,n} = \sum_{v=1}^{a-1} \rho_v s_v^{-n-1} = a^{-1} \cdot 2\sqrt{pq} (\sqrt{q/p})^z \sum_{v=1}^{a-1} s_v^{-n+1} \cdot \sin \phi_v \cdot \sin z\phi_v$$

and hence finally

$$u_{z,n} = a^{-1} 2^n p^{(n-z)/2} q^{(n+z)/2} \sum_{v=1}^{a-1} \cos^{n-1} \frac{\pi v}{a} \sin \frac{\pi v}{a} \sin \frac{\pi z v}{a} \quad (54)$$

This, then, is an explicit formula for the probability of ruin at the n^{th} trial. It goes back to Lagrange and has been derived by classical authors in various ways, but it continues to be rediscovered in the modern literature.

Passing to the limit as $a \rightarrow \infty$ we get the probability that in a game against an infinitely rich adversary a player with initial capital z will be ruined at the n^{th} trial.

A glance at the sum in (54) shows that the terms corresponding to the summation indices $v = k$ and $v = a - k$ are of the same absolute value; they are of the same sign when n and z are of the same parity and cancel otherwise. Accordingly $u_{z,n} = 0$ when $n - z$ is odd while for even $n - z$ and $n > 1$

$$u_{z,n} = a^{-1} 2^{n+1} p^{(n-z)/2} q^{(n+z)/2} \sum_{v < a/2} \cos^{n-1} \frac{\pi v}{a} \sin \frac{\pi v}{a} \sin \frac{\pi z v}{a} \quad (55)$$

the summation extending over the positive integers $< a/2$. This form is more natural than (54) because now the coefficients $\cos \pi v/a$ form a decreasing sequence and so for large n it is essentially only the first term that counts.

2.6 Connection with Diffusion Processes

This section is devoted to an informal discussion of random walks in which the length δ of the individual steps is small but the steps are spaced so close in time that the resultant change appears practically as a continuous motion. A passage to the limit leads to the Wiener process (Brownian motion) and other diffusion processes. The intimate connection between such processes and random walks greatly contributes to the understanding of both. The problem may be formulated in mathematical as well as in physical terms.

It is best to begin with an *unrestricted random walk* starting at the origin. The n^{th} step takes the particle to the position \mathbf{S}_n where $\mathbf{S}_n = \mathbf{X}_1 + \cdots + \mathbf{X}_n$ is the sum of n independent random variables each assuming the values $+1$ and -1 with probabilities p and q , respectively. Thus

$$\mathbf{E}(\mathbf{S}_n) = (p - q)n \quad , \quad Var(\mathbf{S}_n) = 4pqn \quad (56)$$

The figure below presents the first 10,000 steps of such a random walk with $p = q = \frac{1}{2}$; to fit the graph to a printed page it was necessary to choose appropriate scales for the two axes. Let us now go a step further and contemplate a motion picture of the random walk. Suppose that it is to fake 1000 seconds (between 16 and 17 minutes). To present one million steps it is necessary that the random walk proceeds at the rate of one step per millisecond, and this fixes the time scale. What units are we to choose to be reasonably sure that the record will fit a screen of a given height?

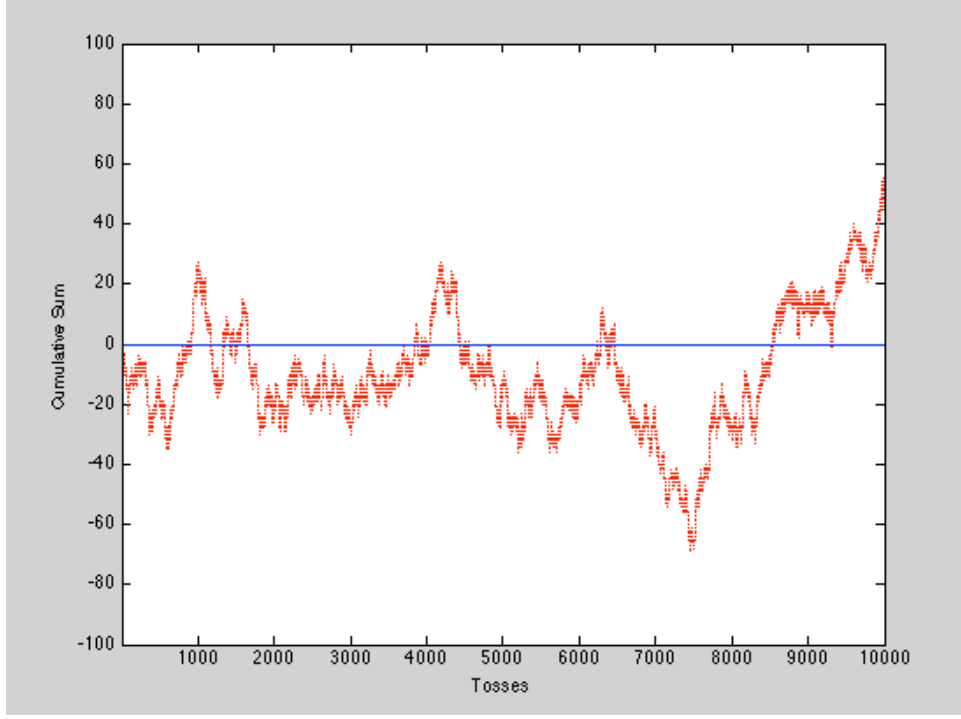


Figure 1: The record of 10,000 tosses of an ideal coin

For this question we use a fixed unit of measurement, say inches or feet, both for the screen and the length of the individual steps. We are then no longer concerned with the variables \mathbf{S}_n , but with $\delta\mathbf{S}_n$, where δ stands for the length of the individual steps. Now

$$\mathbf{E}(\mathbf{S}_n) = (p - q)\delta n \quad , \quad \text{Var}(\mathbf{S}_n) = 4pq\delta^2 n \quad (57)$$

and it is clear from the central limit theorem that the contemplated film is possible only if for $n = 1,000,000$ both quantities in (657) are smaller than the width of the screen. But if $p = q$ and δn is comparable to the width of the screen, $\delta^2 n$ will be indistinguishable from 0 and the film will show linear motion without visible chance fluctuations. The character of the random walk can be discerned only when $\delta^2 n$ is of a moderate positive magnitude, and this is possible only when $p - q$ is of a magnitude comparable to δ .

If the question were purely mathematical we should conclude that the desired graphical presentation is impossible unless $p = q$, but the situation is entirely

different when viewed from a physical point of view. In Brownian motion we see particles suspended in a liquid moving in random fashion, and the question arises naturally whether the motion can be interpreted as the result of a tremendous number of collisions with smaller particles in the liquid. It is, of course, an over-simplification to assume that the collisions are spaced uniformly in time and that each collision causes a displacement precisely equal to $\pm\delta$. Anyhow, for a first orientation we treat the impacts as governed by the Bernoulli-type trials we have been discussing and ask whether the observed motion of the particles is compatible with this picture. From actual observations we find the average displacement c and the variance D for a unit time interval. Denote by r the (unknown) number of collisions per time unit. Then we must have, approximately,

$$(p - q)\delta r = c \quad , \quad 4pq\delta^2 r = D \quad (58)$$

In a simulated experiment no chance fluctuations would be observable unless the two conditions (58) are satisfied with $D > 0$. An experiment with $p = 0.6$ and $\delta r = 1$ is imaginable, but in it the variance would be so small that the motion would appear deterministic: A clump of particles initially close together would remain together as if it were a rigid body.

Essentially the same consideration applies to many other phenomena in physics, economics, learning theory, evolution theory, etc., when slow fluctuations of the state of a system are interpreted as the result of a huge number of successive small changes due to random impacts. The simple random-walk model does not appear realistic in any particular case, but fortunately the situation is similar to that in the central limit theorem. Under surprisingly mild conditions the nature of the individual changes is not important, because the observable effect depends only on their expectation and variance. In such circumstances it is natural to take the simple random-walk model as universal prototype.

To summarize, as a preparation for a more profound study of various stochastic processes it is natural to consider random walks in which the length δ of the individual steps is small, the number r of steps per time unit is large, and $p - q$ is small, the balance being such that (58) holds (where c and $D > 0$ are given constants). The words large and small are vague and must remain flexible for practical applications? (The number of molecular shocks per time unit is beyond imagination. At the other extreme, in evolution theory one

considers small changes from one generation to the next, and the time separating two generations is not small by everyday standards. The number of generations considered is not fantastic either, but may go into many thousands. The point is that the process proceeds on a scale where the changes appear in practice continuous and a diffusion model with continuous time is preferable to the random-walk model.) The analytical formulation of the problem is as follows. To every choice of δ , r , and p there corresponds a random walk. We ask what happens in the limit when $\delta \rightarrow 0$, $r \rightarrow \infty$, and $p \rightarrow \frac{1}{2}$ in such a manner that,

$$(p - q)\delta r \rightarrow c \quad , \quad 4pq\delta^2 r \rightarrow D \quad (59)$$

The main procedure is to start from the difference equations governing the random walks and the derivation of the limiting differential equations. It turns out that these differential equations govern well defined stochastic processes depending on a continuous time parameter. The same is true of various obvious generalizations of these differential equations, and so this method leads to the important general class of diffusion processes.

We denote by $\{\mathbf{S}_n\}$ the standard random walk with unit steps and put

$$v_{k,n} = \mathbf{P}\{\mathbf{S}_n = k\} \quad (60)$$

In our random walk the n^{th} step takes place at epoch n/r , and the position is $\mathbf{S}_n\delta = k\delta$. We are interested in the probability of finding the particle at a given epoch t in the neighborhood of a given point x , and so we must investigate the asymptotic behavior of $v_{k,n}$ when $k \rightarrow \infty$ and $n \rightarrow \infty$ in such a manner that $n/r \rightarrow t$ and $k\delta \rightarrow x$. The event $\{\mathbf{S}_n = k\}$ requires that n and k be of the same parity and takes place when exactly $(n + k)/2$ among the first n steps lead to the right.

The approach based on the appropriate difference equations is very interesting. Considering the position of the particle at the n^{th} and the $(n + 1)^{st}$ trial it is obvious that the probabilities $v_{k,n}$ satisfy the difference equations

$$v_{k,n+1} = pv_{k-1,n} + qv_{k+1,n} \quad (61)$$

Now $v_{k,n}$ is the probability of finding $\mathbf{S}_n\delta$ between $k\delta$ and $(k + 2)\delta$, and since the interval has length 2δ we can say that the ratio $v_{k,n}/(2\delta)$ measures locally the probability per unit length, that is the probability *density* and

$$v_{k,n}/(2\delta) \rightarrow v(t, x).$$

On multiplying by 2δ it follows that $v(t, x)$ should be an *approximate* solution of the difference equation

$$v(t + r^{-1}, x) = pv(t, x - \delta) + qv(t, x + \delta) \quad (62)$$

Since v has continuous derivatives we can expand the terms according to Taylor's theorem. Using the first-order approximation on the left and second-order approximation on the right we get (after canceling the leading terms)

$$\frac{\partial v(t, x)}{\partial t} = (q - p)\delta r \cdot \frac{\partial v(t, x)}{\partial x} + \frac{1}{2}\delta^2 r \frac{\partial^2 v(t, x)}{\partial x^2} + \dots \quad (63)$$

In our passage to the limit the omitted terms tend to zero and (63) becomes in the limit

$$\frac{\partial v(t, x)}{\partial t} = -c \frac{\partial v(t, x)}{\partial x} + \frac{1}{2}D \frac{\partial^2 v(t, x)}{\partial x^2} \quad (64)$$

This is a special diffusion equation also known as the Fokker-Planck equation for diffusion. Now the function v given by

$$v(t, x) = \frac{1}{\sqrt{2\pi Dt}} e^{-\frac{1}{2}(x-ct)^2/Dt} \quad (65)$$

satisfies the differential equation (63). Furthermore, it can be shown that (64) represents the only solution of the diffusion equation having the properties required by a probabilistic interpretation.

Thus, we have demonstrated some brief and heuristic indications of the connections between random walks and general diffusion theory.

As another example let take the ruin probabilities $u_{z,n}$ discussed earlier. The underlying difference equations (35) differ from (61) in that the coefficients p and q are interchanged. The formal calculations indicated in (63) now lead to a diffusion equation obtained from (64) on replacing $-c$ by c . Our limiting procedure leads from the probabilities $u_{z,n}$ to a function $u(t, \xi)$ which satisfies this modified diffusion equation and which has probabilistic significance similar to $u_{z,n}$: In a diffusion process starting at the point $\xi > 0$ the probability that the particle reaches the origin *before* reaching the point $\alpha > \xi$ and that this event occurs in the time interval $t_1 < t < t_2$ is given by the integral of

$u(t, \xi)$ over this interval.

The formal calculations are as follows. For $u_{z,n}$ we have the explicit expression (55). Since z and n must be of the same parity, $u_{z,n}$ corresponds to the interval between n/r and $(n+2)/r$, and we have to calculate the limit of the ratio $u_{z,n}r/2$ when $r \rightarrow \infty$ and $\delta \rightarrow 0$ in accordance with (59). The length a of the interval and the initial position z must be adjusted so as to obtain the limits α and ξ . Thus $z \sim \xi/\delta$ and $a \sim \alpha/\delta$. It is now easy to find the limits for the individual factors in (55).

From (59) we get $2p \sim 1 + c\delta/D$, and

$$2q \sim 1 - c\delta/D$$

From the second relation in (59) we see that $\delta^2 r \rightarrow D$. Therefore

$$\begin{aligned} (4pq)^{1/2} (q/p)^{z/2} &\sim (1 - c^2 \delta^2 / D^2)^{(rt)/2} (1 - 2c\delta/D)^{\xi/(2\delta)} \\ &\sim e^{-\frac{1}{2}t/D} \cdot e^{-c\xi/D} \end{aligned} \quad (66)$$

Similarly for fixed v

$$\left(\cos \frac{v\pi\delta}{\alpha} \right)^n \sim \left(1 - \frac{v^2\pi^2\delta^2}{2\alpha^2} \right)^{tr} \sim e^{-\frac{1}{2}v^2\pi^2 Dt/\alpha^2} \quad (67)$$

Finally $\sin v\pi\delta/\alpha \sim v\pi\delta/\alpha$. Substitution into (55) leads formally to

$$u(t, \xi) = \pi D \alpha^{-2} e^{-\frac{1}{2}(ct+2\xi)c/D} \sum_{v=1}^{\infty} v e^{-\frac{1}{2}v^2\pi^2 Dt/\alpha^2} \sin \frac{\pi\xi v}{\alpha} \quad (68)$$

(Since the series converges uniformly it is not difficult to justify the formal calculations.) In physical diffusion theory (68) is known as *Fürth's formula for first passages*.