How Einstein and/or Schrödinger should have discovered Bell's Theorem in 1936

Terry Rudolph

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This note shows how one can be led from considerations of quantum steering to Bell's theorem. The point is that steering remote systems by choosing between two measurements can be described in a local theory if we take quantum states to be associated many-to-one with the underlying "real states" of the world. Once one adds a third measurement this is no longer possible. Historically this is not how Bell's theorem arose - there are slight and subtle differences in the arguments - but it could have been.

Afterword

Following is the appendix of an incomplete paper from mid-2003[1], that I completely forgot existed until a bit over a year ago when a very nice talk by Howard Wiseman[2] triggered me into searching through old notes for my vaguely recollected version of "Bell's theorem via steering". The somewhat long full paper titled *Quassical Mechanics* is incomplete, it primarily contains a variety of examples of "toy theories" following the ideas of Rob Spekkens [3]. One of them (eventually!) led to, and was superseded by, arXiv:1111.5057. Having given up on myself getting around to completing it anytime soon, but having had a discussion with Reinhard Werner the week before last during which he expressed the opinion that 'Einstein should have discovered Bell's theorem via steering', I'm posting this particular part of it as-is. The simple structure of the argument has not quite been captured yet by recent work on steering and nonlocality (primarily by Wiseman and colleagues[4]).

Basically the appendix is about how, what we would now call a "ψ-epistemic" interpretation of quantum states (following Harrigan and Spekkens[5], can be used to save locality when one considers steering the remote quantum state of a system using only *two* measurements, as was done in the EPR paper. However, as soon as one adds a generic third measurement, locality cannot be saved. This seems to contradict the well known fact that CHSH inequality violation only requires a choice between two measurements. But that argument actually relies on looking at correlations of the two measurements

[1]The only changes I have made are to add references and change 1932 to 1936!

[2]Based primarily around H. Wiseman, Contemporary Physics 47, 79-88 (2006); quant-ph/0509061

[3]W. Spekkens, Phys. Rev. A 75, 032110 (2007); quant-ph/0401052'

[4]H. M. Wiseman, S. J. Jones, and A. C. Doherty, Phys. Rev. Lett. 98, 140402 (2007)

[5]N. Harrigan, R.W. Spekkens, Found. Phys. 40, 125 (2010);quant-ph/0706.2661.

with a pair of measurements at the remote system as well. The argument I'm interested in here is about what just one party can infer about the "real state" of affairs at the remote system necessarily being changed ("steered") nonlocally based solely on their ability to steer its quantum state by, in this case, one of *three* different measurements. Why I like it is that one never talks about the real state of affairs (the "on tic state") of the system being measured to do the steering.

The origins of my thinking at all about classical versus quantum steering go back to working with Rob Spekkens on two party cryptography[6][7] where steering plays a crucial role, and much of my thinking was influenced of course by discussions with him. After he sent me his first ideas about and proofs of preparation contextuality[8] I simplified them based around what I knew from this simple nonlocality proof, and conversely in this version below I mention preparation non-contextuality as being the constraint that locality imposes for this style of argument. However; as far as I can see the precise and interesting connections between proofs of nonlocality and proofs of preparation contextuality have still not been completely fleshed out, though Barrett (private communication) has made some progress in this regard. **APPENDIX A of incomplete article Quassical Mechanics - draft of July 29, 2003**

A How Einstein and/or Schrödinger should have discovered Bell?s theorem in 1936

Bell's theorem - the empirical fact that features of this universe cannot be described by a local theory - is a statement of physics which transcends merely quantum mechanics. Bell's theorem is the only facet of quantum mechanics I believe will still be considered a fascinating insight into nature in a few hundred years time.

In this appendix I will attempt a little revisionist history. In particular, I will attempt to show how a very simple argument establishing the impossibility of a local hidden variable (LHV) description of QM was lingering on the edge

[6] R.W. Spekkens and T. Rudolph, Quantum Inform. Compu. 2, 66 (2002); quant-ph/0107042.

[7] T. Rudolph and R.W. Spekkens, Phys. Rev. A 70, 052306 (2004); quant-ph/0310060

[8] R.W. Spekkens, Phys. Rev. A 71, 052108 (2005); quant-ph/0406166

of Schrödinger's and Einstein's consciousness in 1936. In particular, in 1936 the two of them, via various correspondences [1], were collectively considering the following features of QM:

- *The quantum mechanical wavefunction may not be a complete description.* The possibility that the wave-function was an epistemic "catalog of information" was under consideration.
- *The possibility of steering.* Inspired by the EPR paper, Schrödinger had proven the quantum steering theorem, in large (though not complete) generality.

Both knew that if pure quantum states are taken to be states of reality, then the possibility of steering is violently incompatible with locality. In fact, the term 'steering' was chosen by Schrödinger precisely to reflect this fact - in such scenarios it seems that an action performed on one half of an entangled system nonlocally "steers" or "drives" the wavefunction of the other system[9]. The purpose of this section is to show how, by a simple argument, this conceptual incompatibility could have been proven algebraically to hold for all LHV theories, thereby establishing what we know today as Bell's theorem.

The quantum steering theorem is [2]:

Theorem: *Given an entangled state $|\psi_{AB}\rangle$ of two systems A, B, a measurement on system A can collapse system B to the set of states $\{|\phi_i\rangle\}$ with associated probabilities p_i, if and only if*

$$\rho_B = \sum_i p_i |\phi_i\rangle \langle\phi_i|$$

where $\rho_B \equiv Tr_A |\psi_{AB}\rangle \langle\psi_{AB}|$ is the reduced state of system B. Schrödinger in fact only proved the theorem for ensembles of states $|\phi_i\rangle$ which are linearly independent (possibly non- orthogonal); this is more than we will need here.

In examining the description of steering in a local hidden variable theory, we presume that the actual physical properties of system B are described

[9]At the end of his paper on steering Schrödinger mused that perhaps the resolution would be found in a certain dephasing process (known today as 'decoherence') which prevents us from creating spatially separated entangled states in practice. This has turned out not to be the case.

by a complete set of variables λ. No claims are made about the specific nature of these variables, other than they should correctly reproduce the predictions of QM. This entails certain restrictions. For instance, consider a von-Neumann measurement described in QM by the two projection operators $|\chi\rangle\langle\chi|$, $I-|\chi\rangle\langle\chi|$. We know that there is a state $|\chi\rangle$ of the system, which gives one measurement outcome with certainty, the other with probability zero. Since the measurement outcomes are presumed to be dictated by the particular value of λ governing the physics of the system, we see that the set of all possible λ or the system contains at least two disjoint sets - a set of those values which yield outcome $|\chi\rangle\langle\chi|$ with certainty and those which yield $I-|\chi\rangle\langle\chi|$ with certainty. (There could in general be values of λ which lead to neither outcome with certainty). We denote by S_χ the subset of λ values which lead to outcome χ with probability 1.

In a steering scenario, system B is described quantum mechanically by the mixed state ρ_B. We know that this state can be steered to the eigenstates of ρ_B, which are orthogonal. Since each of these eigenstates are associated with disjoint values of λ, we see that, under a presumption of locality, ρ_B must be associated with a probabilistic distribution over at least two different λ. We denote the set of all λ underlying ρ_B by S_ρ, and denote by $\nu(\lambda)$ any distribution over S_ρ that is the 'hidden variable' description of B. The presumption of locality also indicates that a measurement on system A cannot change the 'real state of affairs' at B - in particular, therefore, it cannot change the value of λ governing B, and thus $\nu(\lambda)$, which is used by the observer at B to describe their system, is unaffected by the measurement performed at A. For simplicity, from here on we limit ourselves to the case where ρ_B is two-dimensional, and further we will take $\rho_B = I/2$. that is, the maximally mixed state.

Let us first formalize the reasoning of Schrödinger and Einstein, which yields a simple argument against local hidden variables if pure quantum states are 'state of reality'. More precisely, we examine the possibility that pure states are ontic - they correspond to a definite value of λ, while mixed quantum states are epistemic - they correspond to a distribution over some λ. Thus, in the ontic view, the state $|x\rangle$ actually corresponds to some specific value $\lambda_x \in S_x$, we therefore associate $|x\rangle$ with a delta function distribution $\delta(\lambda_x)$ over the hidden variables. We need only consider the case where steering is performed either to a pair of orthogonal states $|x\rangle, |X\rangle$ or to another pair of orthogonal states $|y\rangle, |Y\rangle$, with $0 < |\langle x|y\rangle|^2 < 1$. That is,

$$\rho_B = \tfrac{1}{2}|x\rangle\langle x| + \tfrac{1}{2}|X\rangle\langle X| = \tfrac{1}{2}|y\rangle\langle y| + \tfrac{1}{2}|Y\rangle\langle Y|$$

Locality ensures that $S_{\rho_B} = S_x \cup S_X = S_y \cup S_Y$. (Thus all values of $\lambda \in S_\rho$ would yield one of the measurement outcomes $|x\rangle\langle x|, |X\rangle\langle X|, |y\rangle\langle y|$, or $|Y\rangle\langle Y|$ with certainty.) However, the crucial use of locality is to enforce *preparation non-contextuality* [3]. That is, regardless of questions of locality, in order for an ontic interpretation of pure states to be consistent, it is necessary that two different preparation procedures leading to the same mixed state are actually described by different distributions over the hidden variables. For example, in this case, one needs that $\frac{1}{2}\delta(\lambda_x) + \frac{1}{2}\delta(\lambda_X) = \nu_1(\lambda)$, while $\frac{1}{2}\delta(\lambda_y) + \frac{1}{2}\delta(\lambda_Y) = \nu_2(\lambda)$, where the two distributions $\nu_1(\lambda), \nu_2(\lambda)$ are both valid hidden variable descriptions of ρ_B. This requirement shows that the procedure for preparing ρ_B is necessarily contextual in such an interpretation. In the steering scenario, the initial distribution $\nu(\lambda)$ s unaffected by the measurement at A. Hence the role of locality is to enforce $\nu_1 = \nu_2$, which then implies

$$\nu(\lambda) = \tfrac{1}{2}\delta(\lambda_x) + \tfrac{1}{2}\delta(\lambda_X) = \tfrac{1}{2}\delta(\lambda_y) + \tfrac{1}{2}\delta(\lambda_Y)$$

Such a description is inconsistent, by virtue of the fact that within the ontic view we necessarily have $\lambda_x \neq \lambda_X \neq \lambda_y \neq \lambda_Y$. Such an argument contains the essence of what disturbed Einstein and Schrödinger, in a slightly complicated form.

If pure quantum states are epistemic, however, we must go a little further in order to rule out local hidden variable theories. Under the epistemic view, the process of steering is simply reflects the change in information that the observer holding system A has about the system B, based upon their measurement outcome on system A. The particular correlation between A and B is presumed known of course. As we have seen, steering appears in some form both classically and quassically, which are local physical descriptions. Quassically we even obtain steering to multiple different pure state decompositions. However, the argument below shows that quassically we cannot simulate all such steering scenarios.

Let us use the notation that $x(\lambda)$ denotes the distribution over S_x corresponding to the state $|x\rangle$. As mentioned, locality ensures that the distribution $\nu(\lambda)$ is not affected by the measurement at A. Clearly we must have

$$\nu(\lambda) = \tfrac{1}{2}x(\lambda) + \tfrac{1}{2}X(\lambda) = \tfrac{1}{2}y(\lambda) + \tfrac{1}{2}Y(\lambda) \qquad (1)$$

Normalization relations of the form $\int_{S_x} d\lambda\, x(\lambda)$ must be satisfied. The distributions $x(\lambda), y(\lambda)$ cannot be disjoint (since if $S_y \subset S_X$ then the probability of obtaining an outcome $|x\rangle\langle x|$ when a system is in the state $|y\rangle$ would be zero). That is, there is an overlap between the regions S_x and S_y, which we denote S_1. Note that values of λ in this region yield measurement outcomes $|x\rangle\langle x|$ and $|y\rangle\langle y|$ with certainty - the nonorthogonality of $|x\rangle, |y\rangle$ is reflected in the fact that the distribution $y(\lambda)$ only partially overlaps S_x. More precisely, in order to conform with the predictions of QM, we must have that

$$\int_{S_x} d\lambda\, y(\lambda) = \int_{S_1} d\lambda\, y(\lambda) = |\langle x\,|\,y\rangle|^2 \equiv \alpha \tag{2}$$

In fact there are 4 disjoint regions of the λ-space to consider: $S_1 \equiv S_X \subset S_y$, $S_2 \equiv S_x \subset S_Y$, $S_3 \equiv S_X \subset S_y$, $S_4 \equiv S_X \subset S_Y$. We will use the notation that

$$x_j \equiv \int_{S_j} d\lambda\, x(\lambda) \quad , \quad j = 1, ..., 4$$

and so on.

Clearly, by integrating (1) over the appropriate regions, we have the following constraints:

$$\nu_j = \tfrac{1}{2}x_j + \tfrac{1}{2}X_j = \tfrac{1}{2}y_j + \tfrac{1}{2}Y_j \quad , \quad j = 1, ..., 4 \tag{3}$$

From equations of the form (2) we obtain

$$x_1 = y_1 = X_4 = Y_4 = \alpha$$

$$x_2 = y_3 = X_3 = Y_2 = 1 - \alpha$$

with all other values equal to 0. Thus, by (3), $\nu_1 = \nu_4 = \alpha/2$, while $\nu_2 = \nu_3 = (1-\alpha)/2$.

In order to obtain a contradiction, we need to consider a third pair of orthogonal states $|z\rangle, |Z\rangle$ which, by the steering theorem, can also be steered to via a measurement on A. For simplicity, we presume that the state $|z\rangle$ 'bisects' (has equal overlap with) the states $|x\rangle, |y\rangle$. Thus

$$|\langle z\,|\,x\rangle|^2 = |\langle z\,|\,y\rangle|^2 = |\langle Z\,|\,X\rangle|^2 = |\langle Z\,|\,Y\rangle|^2 \equiv \beta = \tfrac{1}{2}(1 + \sqrt{\alpha})$$

the last term being the quantum mechanical prediction. From this we deduce that

$$z_1 + z_2 = \beta = z_1 + z_3 \;\; , \;\; z_3 + z_4 = 1 - \beta = z_2 + z_4 \tag{4}$$

$$Z_3 + Z_4 = \beta = Z_2 + Z_4 \ , \ \ Z_1 + Z_2 = 1 - \beta = Z_1 + Z_3 \tag{5}$$

Clearly $z_2 = z_3$ and $Z_2 = Z_3$. We must also have

$$\nu_j = \tfrac{1}{2} z_j + \tfrac{1}{2} Z_j \quad , \quad j = 1, ..., 4$$

There is no way to satisfy all these equations, subject to the necessary requirement $z_j, Z_j \geq 0$. For example, an independent set of the above equations is

$$z_1 + z_2 = Z_2 + Z_4 = \beta \tag{6}$$

$$z_2 + z_4 = Z_1 + Z_2 = 1 - \beta \tag{7}$$

$$z_1 + Z_1 = \alpha \tag{8}$$

From these we get

$$Z_1 = \alpha - z_1 = \alpha - (\beta - z_2) = \alpha - \beta + (1 - \beta - z_4) = 1 - 2\beta + \alpha - z_4$$

which, using $\beta = \frac{1}{2}(1+\sqrt{\alpha})$, gives $Z_1 = \alpha - \sqrt{\alpha} - z_4$. This is manifestly negative for any $0 \leq \alpha,\ z_4 \leq 1$. This completes the demonstration of incompatibility between local realism and QM.

Although this proof is algebraic and thus reminiscent of GHZ type proofs against local realism, it is in fact more or less equivalent to Mermin?s exposition of Bell inequalities in [4].

References

1 A. Einstein, Letter to Schrödinger (1935). Translation from D. Howard, Stud. Hist. Phil. Sci., 16, 171 (1985).

2 E. Schrödinger. Proc. Camb. Phil. Soc., 31, 555 (1935); Proc. Camb. Phil. Soc., 32, 446 (1936).

3 R. W. Spekkens, Phys. Rev. A, 71, 052108 (2005), arXiv:quant-ph/0406166.

4 N.D. Mermin, Am. J. Phys 49, 940 (1981)