How the result of a single coin toss can turn out to be 100 heads

Ferrie and Combes

March 10, 2014

Abstract

We show that the phenomenon of anomalous weak values is not limited to quantum theory. In particular, we show that the same features occur in a simple model of a coin subject to a form of classical backaction with pre- and post-selection. This provides evidence that weak values are not inherently quantum, but rather a purely statistical feature of pre- and post-selection with disturbance.

In many quantum mechanical experiments, we observe a dissonance between what actually happens and what ought to happen given naïve classical intuition. For example, we would say that a particle cannot pass through a potential barrier-it is *not allowed* classically. In a quantum mechanical experiment the "particle" can "tunnel" through a potential barrier-and a paradox is born. Most researchers spent the 20th century ignoring such paradoxes (that is, "shutting up and calculating" [1]) while a smaller group tried to understand these paradoxes [2-5] and put them to work [6].

The typical way experimentalists probe the quantum world is through measuring the expectation value of an observable A. After many experimental trials the expected value is

$$\langle A \rangle_\psi = \langle \psi | A | \psi \rangle \tag{1}$$

where $|\psi\rangle$ is the quantum state of the system under consideration. The measurement of such an expected value allows us to demonstrate, for example, that Bell's inequalities [5] are violated. Thus measurement of the expected value can have foundational significance.

Now, the potential values one can observe are limited to the eigenvalue range of A. It was surprising, then, that Aharonov, Albert, and Vaidman [7] claimed the opposite. In 1988, they proposed the *weak value* of an observable. The weak value of A is defined as [7, 8]

$$a_w = \frac{\langle \phi | A | \psi \rangle}{\langle \phi | \psi \rangle} \tag{2}$$

where $|\psi\rangle$ and $|\phi\rangle$ re called *pre-* and *post*-selected states. Notice that when $\langle \phi | \psi \rangle$ is close to zero, a_w can lie far outside the range of eigenvalues of A hence the title of [7]: "How the result of a measurement of a component of

the spin of a spin-1/2 particle can turn out to be 100". When this is the case, the weak value is termed *anomalous*.

Weak values are said to have both foundational and practical significance. On one hand, they are claimed to solve quantum paradoxes [9], while on the other, they are claimed to amplify small signals to enhance quantum metrology [10] (but compare to [11-16]). One research program in the weak value community is to examine a paradoxical quantum effect or experiment and then calculate the weak value for that situation. Often the calculated weak value is anomalous. From this we are sup- posed to conclude the paradox is resolved (see, for example, [17] for a recent review). So it would further seem, then, that anomalous weak values, if not *the* source of quantum mysteries, provide deep insight into finding it. Indeed, since their inception, weak values have inspired deep thinking and debate about the interpretation and foundational significance of weak values [18-23].

In this letter we provide a simple classical model which shows anomalous weak values are not limited to quantum theory. In particular, we show the same phenomenon manifests in even the simplest classical system: a coin. This shows that the effect is an artifact of toying with classical statistics and disturbance rather than a physically observable phenomenon.

Let us begin by defining the weak value as it was formally introduced before casting it into a more general picture. We have a system with observable $A = \sum_a a|Keta\langle a|$ and meter system with conjugate variables Q and P so that $[Q, P] = i$. The system and meter start in states $|\psi\rangle$ and

$$|\Phi\rangle = frac1(2\pi\sigma^2)^{1/4} \int dq' \, e^{(-q'^2/4\sigma^2)} \, |q'\rangle$$

interact via the Hamiltonian $H = A \otimes P$, then are measured in the bases $\{|\phi_k\rangle\}$ and $\{|q\rangle\}$ where $q \in [-\infty, \infty]$. We are interested in the joint probability distribution of this measurement:

$$\Pr(q, \phi|\psi, \Phi) = \left|\langle\phi| \langle q| e^{-ixH/\hbar} |\psi\rangle |\Phi\rangle\right|^2$$

where x is a coupling constant. In this case, it can be shown (as in, Chap. 16 of [9]), in the limit $\sigma \to \infty$ [24],

$$\langle\phi| \langle q| e^{-ixA\otimes P/\hbar} |\psi\rangle |\Phi\rangle = \langle\phi|\psi\rangle \, \Phi(q - xa_w) \tag{3}$$

where a_w is the weak value given in (2) and, assuming a_w is real [25], is the average shift of the meter position given the states $|\psi\rangle$ and $|\phi\rangle$. Consider the following example. We take the system observable $A = Z$, the Pauli Z operator, and pre- and post-selected states

$$|\psi\rangle = \cos\frac{\theta}{2}|+1\rangle + \sin\frac{\theta}{2}|-1\rangle \tag{4}$$

$$|\phi\rangle = \cos\frac{\theta}{2}|+1\rangle - \sin\frac{\theta}{2}|-1\rangle \tag{5}$$

where $|+1\rangle$ and $|-1\rangle$ are the $+1$ and -1 eigenstates of Z, respectively. A short calculation reveals

$$a_w == \frac{1}{\cos\theta} \tag{6}$$

thus when $\theta \approx 1.5608$ we have $a_w = 100!$. This is patently non-classical as the states required to observe a value $a_w > 1$, say, are in different bases. Next, we will show how to obtain an anomalous weak value from a system-meter picture and statistical conditioning.

The most general kind of measurement we can perform on a quantum system is described by a set of Kraus operators and their corresponding positive operator valued measure (POVM). To derive the Kraus operators for the above situation we expand the unitary to first order in x:

$$U(x) = e^{-ixA\otimes P/\hbar} \approx \mathbb{I}\otimes\mathbb{I} - ixA\otimes P$$

To this order in perturbation theory, the Kraus operators for a position measurement on the meter are

$$M_q = \langle q|\, U(x)\, |\Phi\rangle = \mathbb{I}\,\langle q\,|\,\Phi\rangle - ixA\,\langle q|\, P\,|\Phi\rangle$$

Using

$$P = -i\hbar\frac{\partial}{\partial q} \text{ and } \partial_y e^{(-y^2/4\sigma^2)} = (-y/42\sigma^2)e^{(-y^2/4\sigma^2)}$$

the Kraus operator becomes

$$M_q = \left[\mathbb{I} - q\frac{x}{\sigma^2}A\right]\Phi(q) \tag{7}$$

As we have specialized to the case where A is a Pauli operator we can use $A^\dagger = A$ and $A^\dagger A = \mathbb{I}$. The POVM elements are $E_q = M_q^\dagger M_q$ which to first order in x are:

$$E_q = |\Phi(q)|^2 \left[\mathbb{I} - q\frac{x}{\sigma^2}A\right] \tag{8}$$

where σ^2 is the initial variance of the Gaussian meter state and x is the coupling constant. If we course grain over the meter measurements so that $q \leq 0$ is identified as outcome "+1" and $q > 0$ is identified as the result "−1", then the effective POVM elements are

$$E_+ = \int_{-\infty}^{0} dq\, E_q \;\text{ and }\; E_- = \int_0^{\infty} dq\, E_q \tag{9}$$

performing the integral gives

$$E_\pm = \tfrac{1}{2}[\mathbb{I} \pm (2x/\sqrt{2\pi\sigma^2})A]$$

Collecting the constants we define $\lambda \equiv 2x/\sqrt{2\pi\sigma^2}$ so that the POVM becomes

$$E_\pm = \frac{1}{2}[\mathbb{I} \pm \lambda A] \tag{10}$$

As $\lambda \to 0$ the measurement approaches the trivial one, conveying no information and leaving the post-measurement unaffected. Identical POVM elements can be obtained from second order perturbation theory. For what follows we find it convenient to introduce a classical random variable, $s \in \{\pm 1\}$ for the sign of the outcome, then the POVM can be written as

$$E_\pm = \frac{1}{2}[\mathbb{I} + s\lambda A] \tag{11}$$

This POVM describes a weak measurement of the operator A in the state $|\psi\rangle$. This can be seen from the probability of observing the outcome s

$$\Pr(s|\psi) = \frac{1}{2}(1 + s\lambda \langle\psi| A |\psi\rangle) \tag{12}$$

which is correlated with the expectation value of the operator A.

We now calculate the conditional expectation of the random variable s given the pre- and post-selected states $|\psi\rangle$ and $|\phi\rangle$ respectively

$$\mathbb{E}_{s|\phi,\psi}[s] = \sum_{s=\pm 1} s\frac{\Pr(s,\phi|\psi)}{\Pr(s|\psi)} = \sum_{s=\pm 1} s\frac{|\langle\phi| E_s |\psi\rangle|^2}{|\langle\phi|\psi\rangle|^2} \tag{13}$$

$$= \sum_{s=\pm 1} \frac{s}{4}\frac{\langle\phi|\mathbb{I} + s\lambda A|\psi\rangle\langle\psi|\mathbb{I} + s\lambda A|\phi\rangle}{|\langle\phi|\psi\rangle|^2} \tag{14}$$

where $\mathbb{E}_{x|y}[f(x)]$ denotes the conditional expectation of $f(x)$ given y. This result can also be arrived at using Bayes rule to determine $\Pr(s, \phi|\psi)$, which is known as the "ABL rule" in quantum theory (after Aharonov, Bergmann, and Lebowitz [26]). Expanding the numerator and neglecting the $O(\lambda^2)$ term we arrive at

$$\mathbb{E}_{s|\phi,\psi}[s] = \sum_{s=\pm 1} \frac{s}{4}\left(1 + 2s\lambda \frac{\langle\phi| A |\psi\rangle}{\langle\phi|\psi\rangle}\right) \tag{15}$$

$$= \lambda \frac{\langle\phi| A |\psi\rangle}{\langle\phi|\psi\rangle} \tag{16}$$

Thus the conditional expectation of s results in a quantity proportional to the weak value. Since the constant of proportionality is λ, to arrive directly at the weak value we consider the conditional expectation of the random variable s/λ. Uisng this, we will show

$$\mathbb{E}_{s|\phi,\psi}\left[\frac{s}{\lambda}\right] \to \frac{\langle\phi| A |\psi\rangle}{\langle\phi|\psi\rangle} \text{ as } \lambda \to 0$$

Using Eq. (16), we have

$$\mathbb{E}_{s|\phi,\psi}\left[\frac{s}{\lambda}\right] = \frac{1}{\lambda}\mathbb{E}_{s|\phi,\psi}[s] = \frac{\langle\phi| A |\psi\rangle}{\langle\phi|\psi\rangle} + O(\lambda) \tag{17}$$

Now we take a limit to remove the $O(\lambda)$ term to arrive at exactly the weak value [24]

$$a_w = \lim_{\lambda\to 0} \mathbb{E}_{s|\phi,\psi}\left[\frac{s}{\lambda}\right] = \frac{\langle\phi| A |\psi\rangle}{\langle\phi|\psi\rangle} \tag{18}$$

To relate this to the meter picture note that $s/\lambda = \sqrt{2\pi}\sigma s/2x$. Thus the limit $\lambda \to 0$ is identical to $\sigma \to \infty$.

From Eq. (12) we can see that

$$\langle\phi| A |\psi\rangle = \mathbb{E}_{s|\psi}\left[\frac{s}{\lambda}\right] = \sum_s \frac{s}{\lambda}\Pr(s|\psi) \tag{19}$$

By the classical law of total expectation we have:

$$\langle\phi| A |\psi\rangle = \mathbb{E}_{s|\psi}\left[\frac{s}{\lambda}\right] = \mathbb{E}_{\phi|\psi}\left[\mathbb{E}_{s|\phi,\psi}\left[\frac{s}{\lambda}\right]\right] \tag{20}$$

From Eq. (18) we know in the limit $\lambda \to 0$ we can replace $\mathbb{E}_{s|\psi}\left[\frac{s}{\lambda}\right]$ with the weak value

$$\langle\phi|\, A\,|\psi\rangle \to \mathbb{E}_{\phi|\psi}\left[\frac{\langle\phi|\, A\,|\psi\rangle}{\langle\phi\,|\,\psi\rangle}\right] \text{ as } \lambda \to 0 \tag{21}$$

So, the weak value arises close to the way it is often envisioned to-as a condition expectation-but to define it properly, we need to include a renormalization by the weakness parameter λ.

Now we demonstrate that it is possible to find anomalous weak values for pre- and post-selected states in the same basis provided there is classical disturbance. In particular, we take $A = Z$, $|\psi\rangle = |{+1}\rangle$ and $|\phi\rangle = |{-1}\rangle$. Using the POVM effects in Eq. (10), the probability of the outcome of the weak measurement is

$$\Pr(s|\psi = +1) = \frac{1}{2}(1 + s\lambda) \tag{22}$$

Since the measurement is in the same basis as the state, the state is unchanged and the final weak value will not be anomalous. Thus, we must do something more. To simulate the disturbance, we now apply a bit-flip channel which conditionally depends on the strength and outcome of the weak measurement. This is reasonable as one would expect, from quantum measurement theory, that the amount of disturbance should depend on the strength of the measurement. After the channel, the state becomes

$$|{+1}\rangle\langle{+1}| \mapsto (1-p)\,|{+1}\rangle\langle{+1}| + p\,|{-1}\rangle\langle{-1}| \tag{23}$$

where we set the probability of the bit-flip to be

$$p = 1 - \frac{1 - \cos\theta}{1 = s\lambda} \tag{24}$$

and θ is a constant of our choosing. Since the probability, given s, of observing $|{-1}\rangle$ in the final measurement is p, we have

$$\Pr(s, \phi|\psi) = \Pr(\phi|s, \psi)\Pr(s|\psi) = \frac{p}{2}(1 + s\lambda) \tag{25}$$

$$= \frac{1}{2}(\cos\theta + s\lambda) \tag{26}$$

Marginalizing over s, we obtain

$$\Pr(\phi|\psi) = \sum_{s=\pm 1} \Pr(s, \phi|\psi) = \cos\theta \tag{27}$$

We now have all the ingredients to calculate the weak value as defined in Eq. (18):

$$a_w = \lim_{\lambda\to 0} \mathbb{E}_{s|\phi,\psi}\left[\frac{s}{\lambda}\right] \tag{28}$$

$$= \lim_{\lambda\to 0} \sum_{s=\pm 1} \frac{s}{\lambda} \frac{\Pr(s,\phi|\psi)}{\Pr(\phi|\psi)} \tag{29}$$

$$= \lim_{\lambda\to 0} \sum_{s=\pm 1} \frac{s(\cos\theta + s\lambda)}{2\lambda\cos\theta} \tag{30}$$

$$= \frac{1}{\cos\theta} \tag{31}$$

which is identical to the calculated weak value (6) using pre- and post-selected states taken from differing bases. Since the state here remains in the Z basis at all times, it is essentially classical. To make this point unequivocally clear, we now give an explicitly classical protocol to realize anomalous weak values.

Our example revolves around a coin where the outcome "Heads" is associated with the sign "+1" while "Tails" is associated with the sign "-1". This allows us to compare the analysis above for a quantum coin case (a qubit) and a classical coin. As before we abstract the sign into a random variable s.

An efficient strong measurement of an unbiased coin after a flip will result in an observer measuring and reporting outcome s with probability $\Pr(\text{report } s|\text{prepare } s) = 1$. A classical weak measurement of the sign of a coin $s \in \{\pm 1\}$ means the observer did not properly ascertain if the coin was heads or tails. Such a measurement might arise from an observer not having the time to properly examine the coin or if there was oil on their glasses. We model this by introducing a probability $\Pr(\text{report } s|\text{prepare } s) = 1 - \alpha$ and $\Pr(\text{report } \neg s|\text{prepare } s) = \alpha$. To make the connection with the weak measurement in quantum coin case, see Eq. (11) and Eq. (12), we take

$$\alpha = \frac{1-\lambda}{2}$$

so that

$$\Pr(s|\psi) = \frac{1}{2}(1 - \lambda s\psi) \tag{32}$$

For a coin that starts in heads $\psi = +1$, so

$$\Pr(s|\psi = +1) = \frac{1}{2}(1 = \lambda s)$$

In this case, the physical meaning of λ is clear-it is strength of the correlation between the result s and the preparation ψ.

We now introduce a classical protocol directly analogous to the quantum protocol that produces anomalous weak values. There are two people, Alice and Bob. The protocol is as follows (see also Fig. 1):

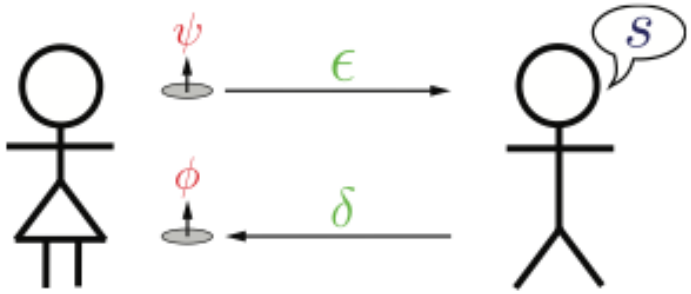

Figure 1: An illustration of the protocol used to realize anomalous classical weak values.

1 **Preselection:** Alice tosses the coin, the outcome ψ is recorded, and she passes it to Bob.

2.a **Weak measurement:** Bob reports s with the probabilities given in equation (32).

2.b **Classical disturbance:** Bob flips the coin with probability given in equation (37) and returns it to Alice.

3 **Postselection:** Alice looks at the coin and records the outcome ϕ.

For concreteness we preselect on heads, that is $\psi = +1$. Bob then makes a weak measurement of the state of the coin, which is described by Eq. (32). In order to implement classical backaction we introduce a probabilistic disturbance parameter δ to our model. The effect of the disturbance on the final measurement is realized through the following correlation between the weak measurement of the sign s and the final strong measurement which we label by ϕ:

$$\Pr(s, \phi = +1 | \psi = +1) = \frac{\delta}{2} \tag{33}$$

$$\Pr(s, \phi = -1 | \psi = +1) = \frac{1}{2}(1 + s\lambda - \delta) \tag{34}$$

This can be compactly written

$$\Pr(s, \phi | \psi = +1) = \frac{1}{2}(1 + \phi\delta + s\lambda)$$

One can easily verify that marginalizing over ϕ reproduces the distribution of s in Eq. (32). Similarly, marginalizing over s yields

$$\Pr(\phi = +1 | \psi = +1) = \delta \tag{35}$$

$$\Pr(\phi = -1 | \psi = +1) = 1 - \delta \tag{36}$$

The point here is Bob will flip the coin (i.e. $+1 \to -1$) with probability $1 - \delta$. Although, δ can be thought of as a "disturbance" parameter, a more entertaining interpretation is to think of Bob as an "λ liar, δ deceiver": Bob accepts the coin and lies about the outcome with probability $\frac{1}{2}(1 - \lambda)$ and then further, to cover his tracks, flips the coin before returning it to Alice with probability $\approx 1 - \delta$. Using Bayes rule we determine the probability that Bob flips or does not flip the coin to be:

$$\Pr(\phi = -1 | \psi = +1) = \frac{1 + s\lambda - \delta}{1 + s\lambda} \quad \text{flip} \tag{37}$$

$$\Pr(\phi = +1 | \psi = +1) = \frac{1 + s\lambda + \delta}{1 + s\lambda} \quad \text{no flip} \tag{38}$$

Now we can ask, does the conditional expectation ever produce an anomalous value? And, indeed it does if we condition on $\phi = -1$:

$$a_w = \lim_{\lambda \to 0} \mathbb{E}_{s|\phi=-1,\psi=+1} \left[\frac{s}{\lambda} \right] \tag{39}$$

$$= \lim_{\lambda \to 0} \sum_{s=\pm 1} \frac{s}{2\lambda} \left(\frac{1 + s\lambda - \delta}{1 - \delta} \right) \tag{40}$$

$$= \frac{1}{1 - \delta} \tag{41}$$

If we change variables to $\delta = 1 - \cos\theta$, we arrive at the same result as the quantum cases in Eqs. (6) and (31). In particular, we see that the classical weak value can be arbitrarily large provided the parameter δ is close to 1 and we pre-select $\psi = +1$ and post-select on $\phi = -1$. Take the example $\delta = 0.99$.

The classical weak value of s, from Eq. (41) with $\delta = 0.99$, is $a_w = 100$. Thus, the outcome of the coin toss is 100 heads!

Before we conclude, we provide some preemptive rebuttals to the criticisms we anticipate our paper will elicit. First, we have pointed out that our model (in fact, any model) requires measurement disturbance for anomalous weak values to manifest. Since, *in theory*, classical measurements can have infinite resolution with no disturbance, some might consider our model non-classical. However, *in practice* classical measurements do have disturbance and do not have infinite precision. While we have not provided a physical mechanism for the disturbance here, it is clear that many can be provided. Thus, we leave the details of such a model open.

The second, and perhaps more significant potential criticism, is that we have given a classical model where only *real* weak values occur. Whereas, the quantum weak value is a complex quantity in general. It is often stated that weak values are "measurable complex quantities" which further allow one to "directly" access other complex quantities [27]. However, the method to "measure" them is to perform separate measurements of the real and imaginary parts. This illustrates that the weak value is actually a *defined* quantity rather than a measured valued. Thus, we can easily introduce complex weak values in our classical model with two observable quantities and simply multiply one by the imaginary unit-not unlike descriptions of circular polarization in classical electromagnetic theory.

In conclusion, our analysis above demonstrates a simple classical model which exhibits anomalous weak values. Recall that the way in which weak values are used in foundational analyses of quantum theory is to show that they obtain anomalous values for "paradoxical" situations. To suggest that this is meaningful or explanatory, it must be the case that such values cannot be obtained classically. Here we have shown they can indeed. Thus, the conclusion that weak values can explain some paradoxical situation or verify its quantumness are called to question. Our results provide evidence that weak values are not inherently quantum, but rather a purely statistical feature of pre- and post-selection with disturbance.

The authors thank Jonathan Gross, Jacob Miller, Howard Wiseman, and Carl Caves for discussions about weak measurements and weak values. This

work was supported in part by National Science Foundation Grant Nos. PHY-1212445 and PHY-1005540, by Office of Naval Research Grant No. N00014-11-1-0082 and by the Canadian Government through the NSERC PDF program.

References

[1] N. D. Mermin, Physics Today 42, 9 (1989).

[2] H. Everett, Ph.D. thesis, Princeton University (1956).

[3] B. S. DeWitt, Physics Today 23, 30 (1970).

[4] D. Bohm, Phys. Rev. 85, 166 (1952).

[5] J. S. Bell, Physics 1, 1, 195 (1964).

[6] D. Deutsch and R. Jozsa, Proc. R. Soc. Lond. A 439, 553 (1992).

[7] Y. Aharonov, D. Z. Albert, and L. Vaidman, Phys. Rev. Lett. 60, 1351 (1988).

[8] I. M. Duck, P. M. Stevenson, and E. C. G. Sudarshan, Phys. Rev. D 40, 2112 (1989).

[9] Y. Aharonov and D. Rohrlich, Quantum Paradoxes (Wiley-VCH, 2005), ISBN 3527403914.

[10] O. Hosten and P. Kwiat, Science 319, 787 (2008).

[11] G. C. Knee, Andrew, S. C. Benjamin, and E. M. Gauger, Phys. Rev. A 87, 012115 (2013).

[12] S. Tanaka and N. Yamamoto, Phys. Rev. A 88, 042116 (2013).

[13] C. Ferrie and J. Combes, Phys. Rev. Lett. 112, 040406 (2014).

[14] G. C. Knee and E. M. Gauger, Phys. Rev. X 4, 011032 (2014).

[15] J. Combes, C. Ferrie, Z. Jiang, and C. M. Caves, arXiv:1309.6620 [quant-ph] (2013).

[16] L. Zhang, A. Datta, and I. A. Walmsley, arXiv:1310.5302 [quant-ph] (2013).

[17] Y. Shikano, Theory of ?Weak Value? and Quantum Mechanical Measurements (InTech, 2012), Chap. 4, ISBN 978-953-51-0058-4.

[18] A. J. Leggett, Phys. Rev. Lett. 62, 2325 (1989).

[19] A. Peres, Phys. Rev. Lett. 62, 2326 (1989).

[20] Y. Aharonov and L. Vaidman, Phys. Rev. Lett. 62, 2327 (1989).

[21] N. D. Mermin, Phys. Rev. Lett. 74, 831 (1995).

[22] M. S. Leifer and R. W. Spekkens, Phys. Rev. Lett. 95, 200405 (2005).

[23] N. D. Mermin, Physics Today 64, 8 (2011).

[24] It will be apparent later that we need to take the limit $\Delta \to \infty$, where $\Delta = \sigma/x$ as x could independently be limited to zero.

[25] We will assume the weak value is real throughout for brevity and comment on complex weak values in the discussion.

[26] Y. Aharonov, P. G. Bergmann, and J. L. Lebowitz, Phys. Rev. 134, B1410 (1964).

[27] J. Dressel, M. Malik, F. M. Miatto, A. N. Jordan, and R. W. Boyd, arXiv:1305.7154 [quant-ph] (2013).